NON-CONSTANT RANK CONSTRAINTS

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Abstract

We study mechanical systems subject to constraint functions that can be dependent at some points and independent at the rest. We discuss how the constraint force can transmit an impulse to the motion at the points of dependence and derive an explicit formula to obtain the "post-impact" momentum in terms of the "pre-impact" momentum.

1 Introduction

In the last years, mechanical systems subject to nonholonomic constraints have received a lot of attention in the literature of Geometric Mechanics (see [2, 3, 7, 4, 5, 9, 13, 14, 16, 17, 19, 20, 24, 26] and references therein). The constraints are usually given by a $2n-k$-dimensional submanifold of the tangent bundle of the configuration manifold. Locally, the constraint manifold can be represented by the annihilation of k independent functions $\{\phi_1, \ldots, \phi_k\}$, where "independent" means that their differentials are linearly independent at each point.

In this paper, we shall consider the more general case in which the constraint functions become dependent at some points. In this first approach to the problem, we shall assume that, globally, the constraints are given by a generalized differentiable codistribution, that is, a codistribution which changes its rank. Many simple examples exhibit this kind of behaviour. For instance, consider a rolling ball on a surface which is rough on some parts but smooth on the rest. On the rough parts, it will roll without slipping and, hence, nonholonomic linear constraints will be present. However, when the sphere reaches a smooth part, the constraints will dissapear.

The first reference in a geometrical context for such kind of systems is [6]. In that paper, the authors were mainly motivated by problems in motion planning. Here, our main concern will be the dynamical laws that govern the motion of the system, so both approaches will be different. In [6], the constraints are provided by a set of global 1-forms on the configuration manifold and, using Frobenius theorem, the authors gave a classification of them in according to the existence of some special sets that can exert a big influence on the trajectories of the system. In particular, the existence of an integral manifold give a sort of partial holonomicity with strong implications.

The approach of considering nonholonomic constraints given by a generalized codistribution D will lead us to define the concepts of regular and singular points, which are slightly different from those in [6]. Indeed, the regular points are those where the codistribution has locally constant rank. In this sense, the generalized codistribution is a regular codistribution on the connected components of the set of regular points. The singular points are those where the codistribution changes its rank. From a dynamical perspective, the situation on the regular points is known: we can derive the equations of motion following d'Alambert principle and treat them using the well-developed theory for nonholonomic lagrangian systems.

However, on the singular points the matter is essentially different. The classical derivation of the equations of motion no longer works and we must solve the problem with other methods. We have adopted a point of view inspired in the theory of impulsive mechanics [1, 10, 11, 12, 15, 21, 22, 23]. Analyzing the trajectories which cross the singular set, we have found that, in certain cases, the constraint force can transmit an impulse to the motion. It is precisely the sudden appearence of new constraints (i.e. the change of rank of the codistribution) which induces this impulsive character. We refer to [8] for a detailed exposition of the results of this paper.

2 Generalized codistributions

We introduce here the notion of a generalized codistribution. All the results in this section are adapted from the ones stated for generalized distributions in [25].

By a **generalized codistribution** we mean a family of linear subspaces $D = \{D_q\}$ of the cotangent spaces T_q^*Q. Such a codistribution is called differentiable if $\forall q \in \operatorname{Dom} D$, there is a finite number of differentiable local 1-forms $\omega_1, ..., \omega_l$ defined on some open neighborhood U of q such that $D_{q'} = \mathrm{span}\{\omega_1(q'), ..., \omega_l(q')\}$ for all $q' \in U$.

We define the rank of D at q as $\rho(q) = \dim D_q$. If D is differentiable, it is clear that ρ is a lower semicontinuous function, since $\rho(q)$ cannot decrease in a neighbourhood of q. If ρ is a constant function, then D is a codistribution in the usual sense.

For a generalized differentiable codistribution D, a point $q \in Q$ will be called **regular** if q is a local maximum of ρ, that is, ρ is constant on an open neighbourhood of q. Otherwise, q will be called a **singular** point of D. The set R of the regular points of D is obviously open. But, in addition, it is dense, since if $q_0 \in S = Q \setminus R$, and U is a neighbourhood of q_0, U necessarily contains regular points of D ($\rho_{|U}$ must have a maximum because it is integer valued and bounded). Consequently, $q_0 \in \bar{R}$. In general R will not be connected.

Remark 2.1 Observe that the notion of singular point defined here is different from the one considered in [6].

Given a generalized codistribution, D, its annihilator, D^o, is the generalized distribution given by

$$\begin{array}{rrcl} D^o: & \mathrm{Dom}\, D \subset Q & \longrightarrow & TQ \\ & q & \longmapsto & D^o_q = (D_q)^o \,. \end{array}$$

Remark that if D is differentiable, D^o is not differentiable, even continuous, in general (the corresponding rank function of D^o will not be lower semicontinuous). In fact, D^o is differentiable if and only if D is a regular codistribution.

We will call M an **integral submanifold** of D if T_mM is annihilated by D_m at each point $m \in M$. M will be an integral submanifold of maximal dimension if

$$T_m M^o = D_m \,,\ \forall m \in M \,.$$

In particular, this implies that the rank of D is constant along M. A **leaf** L of D is a connected integral submanifold of maximal dimension such that every connected integral manifold of maximal dimension of D which intersects L is an open submanifold of L. D will be a **completely integrable** codistribution if for every point $q \in Q$, there exists one leaf passing through q. In that case, the set of leaves defines a general foliation of Q.

3 Impulsive forces

In this section, we discuss classical mechanical systems with impulsive forces [1, 21, 22, 23]. In recent years, there has been some interest on such systems in the context of Geometric Mechanics [10, 11, 12, 15].

Consider a system of n particles in $\mathbb{R}^3$ such that the particle r has mass m_r. We introduce coordinates $(q^{3r-2}, q^{3r-1}, q^{3r})$ for the particle r. Let us denote $F_r = (F^{3r-2}, F^{3r-1}, F^{3r})$ the resultant of all forces acting on the r^{th} particle. The motion of the particle r in an interval $[t, t']$ is determined by the system of integral equations

$$m_r(\dot{q}^k(t') - \dot{q}^k(t)) = \int_t^{t'} F^k(\tau)d\tau \,, \tag{1}$$

where $3r-2 \leq k \leq 3r$. The integrals of the right-hand side are the components of the **impulse** of the force F_r. Equation (1) is a generalized writing of Newton's second law, stated in integral form in order to consider the case of velocities with finite jump discontinuities. This is the case of impulsive forces. If F is impulsive there exists an instant t_0 such that

$$\lim_{t \to t_0} \int_{t_0}^{t} F(\tau)d\tau = P \neq 0 \,. \tag{2}$$

Equation (2) implies that the impulsive force has an infinite magnitude at the point t_0, but we are assuming that its impulse P is well defined and finite.

In the sequel, the velocity vector of the r^{th} particle, $(\dot{q}^{3r-2}, \dot{q}^{3r-1}, \dot{q}^{3r})$, will be denoted by $\dot{q}^r$. Then, the system of integral equations (1) can be written as

$$m_r(\dot{q}^r(t') - \dot{q}^r(t)) = \int_t^{t'} F_r(\tau)d\tau \,.$$

In the presence of given impulsive forces acting on m particles, say, at time t_0, we have

$$\lim_{t \to t_0} \int_{t_0}^{t} F_{r'}(\tau)d\tau = P_{r'} \neq 0 \,,\ 1 \leq r' \leq m \,,$$

and the equation of the impulsive motion is then

$$\sum_{r=1}^{n} \{p_r(t_0)_+ - p_r(t_0)_- - P_r\} \cdot \delta q^r = 0 \,, \tag{3}$$

where $P_r = 0$, $m+1 \leq r \leq n$. We refer to [23] for a detailed derivation.

The impulsive forces may also be caused by constraints, which are termed **impulsive constraints**. In the presence of nonholonomic linear constraints of type $\Psi = 0$, where $\Psi = b_k(q)\dot{q}^k$, the constraint force, $F = F_k\, dq^k$, is given by $F_k = \mu \cdot b_k$, where μ is a Lagrange multiplier. Then the constraint is impulsive if and only if $\lim_{t \to t_0} \int_{t_0}^{t} \mu \cdot b_k d\tau = P_k \neq 0$, for some k.

The presence of such constraints does not invalidate equation (3). It merely means that the virtual displacements δq^r must satisfy certain additional conditions. So, in abscence of impulsive external forces and presence of impulsive constraints, we would have

$$\sum_{r=1}^{n} \Delta p_r(t_0) \cdot \delta q^r = 0 \,, \tag{4}$$

where $\Delta p_r(t_0) = p_r(t_0)_+ - p_r(t_0)_-$.

4 Nonholonomic Lagrangian systems

Let us consider a lagrangian system with regular lagrangian $L : TQ \to \mathbb{R}$, subject to a set of nonholonomic constraints given by a $(2n-m)$-dimensional submanifold M of TQ. M is locally represented by the annihilation of a set of independent functions ϕ_i, for $1 \leq i \leq m$. For simplicity we will assume in the sequel that $\tau_Q(M) = Q$, i.e. the constraints are "purely kinematical" in the sense that they do not impose restrictions on the allowable positions. The motions of the system are forced to take place on M, and this requires the introduction of some "reaction forces". In [16, 17], an intrinsic expression for the equations of motion was obtained, which we will describe below.

To fix notations, let us take $(q^A, \dot{q}^A)$ the bundle coordinates on TQ. Denote by $\Delta = \dot{q}^A \dfrac{\partial}{\partial \dot{q}^A}$ the dilation vector field on TQ and by $S = dq^A \otimes \dfrac{\partial}{\partial \dot{q}^A}$ the canonical vertical endomorphism (see [18]). Then, $\omega_L = -dS^*(dL)$ is the Poincaré-Cartan two-form and $E_L = \Delta(L) - L$ represents the energy of the system. The symplectic form ω_L induces two isomorphisms of $C^\infty(TQ)$-modules (*musical mappings*)

$$\flat_L : \mathfrak{X}(TQ) \longrightarrow \Omega^1(TQ) \,, \; \sharp_L : \Omega^1(TQ) \longrightarrow \mathfrak{X}(TQ) \,,$$

where $\flat_L(X) = i_X \omega_L$ and $\sharp_L = \flat_L^{-1}$. In absence of constraints, the dynamics is given by the solution Γ_L of the equation $i_{\Gamma_L} \omega_L = dE_L$, i.e. $\Gamma_L = \sharp_L(dE_L)$. Indeed, Γ_L is a second order differential equation (SODE) whose solutions are precisely the solutions of the Euler-Lagrange equations for L.

In the presence of constraints, the equations of motion have to be modified as follows. First of all, we define a distribution F on TQ along M by prescribing its annihilator to be a subbundle of T^*TQ which, along the constraint submanifold M, represents the bundle of reaction forces. More precisely, we set $F^o = S^*(TM^o)$.

The equations of motion for the nonholonomic mechanical system are

$$\left\{ \begin{array}{l} (i_X\omega_L - dE_L)_{|M} \in F^o \ , \\ X_{|M} \in TM \ . \end{array} \right. \tag{5}$$

It should be pointed out that each solution of (5) (if there exists one) satisfies automatically the SODE condition along M. Therefore, in local coordinates, the integral curves of X on M are of the form $(q^A(t), \dot{q}^A(t))$, whereby the $q^A(t)$ are solutions of the system of differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = \lambda^i \frac{\partial \phi_i}{\partial \dot{q}^A} \ , \tag{6}$$

together with the constraint equations $\phi_i(q^A, \dot{q}^A) = 0$, and where the λ^i are Lagrange multipliers to be determined.

The nonholonomic system will have a unique solution X if it satisfies $F^{\perp_{\omega_L}} \cap TM = 0$ (*the compatibility condition*). In the case of mechanical systems, this condition is always fulfilled.

5 Mechanical systems subject to generalized constraints

Let us consider a mechanical system with lagrangian function $L : TQ \to \mathbb{R}$, $L(v) = \frac{1}{2}g(v,v) - (U \circ \tau_Q)(v)$, where g is a Riemannian metric on Q and $U : Q \longrightarrow \mathbb{R}$ is the potential energy function. Suppose, in addition, that the system is subject to a set of constraints given by a generalized differentiable codistribution D on Q, that is, we assume that $\tau_Q(D) = Q$. The motions of the system are forced to satisfy the constraints imposed by D.

We know that the codistribution D induces a decomposition of Q into regular and singular points. We write $Q = R \cup S$. Let us fix R_c, a connected component of R. We can consider the restriction of the codistribution to R_c, $D_c = D_{|R_c} : R_c \subset Q \longrightarrow T^*Q$. Obviously, D_c is a regular codistribution, i.e. it has constant rank. Then, let us denote by $D_c^o : R_c \longrightarrow TQ$ its annihilator. Now, we can consider the dynamical problem with regular lagrangian L, subject to the regular codistribution D_c^o and apply the well-developed theory for nonholonomic lagrangian systems [3, 4, 7, 14, 16, 17, 20]. Consequently, our problem is solved on each connected component of R. The situation changes radically if the motion reaches a singular point. The rank of the constraint codistribution can vary suddenly and the classical derivation of the equations of motion for nonholonomic lagrangian systems is no longer valid. Let us explore the behaviour of the system when such a thing occurs.

Consider a trajectory of the system, $q(t)$, which reaches a singular point at time t_0, i.e. $q(t_0) \in S$, such that $q(t_0 - \epsilon, t_0) \subset R$ and $q(t_0, t_0 + \epsilon) \subset R$ for sufficiently small $\epsilon > 0$. The motion along the trajectory $q(t)$ is governed by the following integral writing of Newton's second law. At each component

$$p_A(t') - p_A(t) = \int_t^{t'} F_A(\tau) d\tau \,, \tag{7}$$

on any interval $t \leq t' < \infty$, where F is the resultant of all the forces action on the trajectory $q(t)$. In our case, the unique forces acting are the constraint forces.

	$q(t_0 - \epsilon)$: pre-points	$q(t_0)$: singular point	$q(t_0 + \epsilon)$: post-points
Case 1	$\rho = r$	$\rho_0 = r_0 = r$	$\rho > r$
Case 2	$\rho = r$	$\rho_0 = r_0 < r$	$\rho = r_0$
Case 3	$\rho = r$	$\rho_0 = r_0 < r$	$\rho > r_0$

Table 1: Possible cases

The nature of the force can become impulsive because of the change of rank of the codistribution D. We summarize the situations that can be found in Table 1. On entering the singular set, the rank of the codistribution D at the singular point $q(t_0)$ can be the same as at the preceding points (Case 1) or can be lower (Cases 2 and 3). In these two latter situations, the constraints have collapsed at $q(t_0)$ and this induces a finite jump in the constraint force. As the magnitude of the force is not infinite, there is no abrupt change in the momenta. Then, in all cases, we find no momentum jumps on entering S.

On leaving the singular set, the rank of D at the posterior points can be the same as at $q(t_0)$ (Case 2) or can be higher (Cases 1 and 3). In Case 2 nothing special occurs. In Cases 1 and 3, the trajectory must satisfy, inmediately after the point $q(t_0)$, **additional constraints** which were no present before. In this sense we affirm that the constraint force can become impulsive: if the motion which passes through the singular set and tries to enter the regular one again does not satisfy the new constraints, then it experiments a jump of its momentum due to the presence of the constraint force. In this way, the new values of the momentum will satisfy the constraints. But one has to be careful: the impulsive force will act just on leaving S, on the regular set. Consequently, we must take into account the virtual displacements associated to the posterior regular points. In any case, we will make the convention that the jump happens at $q(t_0)$.

5.1 Momentum jumps

Now, we derive a formula, strongly inspired in the theory of impulsive motion, for the momentum jumps which can occur due to the changes of rank of the codistribution D in Cases 1 and 3.

The key point of the procedure is to take into account the virtual displacements at the regular posterior points, and not those of $q(t_0)$, which are readically different because of the change of rank. From a dynamical perspective, those are the "main" ones, since it is on the regular set where an additional constraint force acts. As we have seen, the momentum jump happens on just leaving S, due to the presence of this additional constraint force R. We refer to [8] for a careful derivation of the proposed equations, as well as for a justification of the cases when there exists a jump.

At $q(t_0)$ we define the following vector subspaces of $T^*_{q(t_0)}Q$

$$
\begin{aligned}
D^-_{q(t_0)} = \{\alpha \in T^*_{q(t_0)}Q \quad / \quad & \exists\tilde{\alpha} : (t_0-\epsilon, t_0) \to T^*Q, \\
& \tilde{\alpha}(t) \in D_{q(t)} \ , \ \lim_{t\to t_0^-} \tilde{\alpha}(t) = \alpha\} \\
D^+_{q(t_0)} = \{\alpha \in T^*_{q(t_0)}Q \quad / \quad & \exists\tilde{\alpha} : (t_0, t_0+\epsilon) \to T^*Q, \\
& \tilde{\alpha}(t) \in D_{q(t)} \ , \ \lim_{t\to t_0^+} \tilde{\alpha}(t) = \alpha\}
\end{aligned}
$$

From the definition of $D^-_{q(t_0)}$ and $D^+_{q(t_0)}$ we have that

$$
(D^-_{q(t_0)})^\perp = \lim_{t\to t_0^-} (D_{q(t)})^\perp \quad \text{and} \quad (D^+_{q(t_0)})^\perp = \lim_{t\to t_0^+} (D_{q(t)})^\perp
$$

where $^\perp$ denotes the orthogonal with respect to the metric g, and the limits $(D^\perp)^-$ and $(D^\perp)^+$ are defined as in the case of D^- and D^+. Since D is a differentiable codistribution then $D_{q(t_0)} \subseteq D^-_{q(t_0)}$ and $D_{q(t_0)} \subseteq D^+_{q(t_0)}$.

We will deduce the existence of jump of momenta depending on the relation between $D^-_{q(t_0)}$ and $D^+_{q(t_0)}$. The possible cases are stated in Table 2.

$D^+_{q(t_0)} \subseteq D^-_{q(t_0)}$	there is no jump of momenta
$D^+_{q(t_0)} \nsubseteq D^-_{q(t_0)}$	possibility of jump of momenta

Table 2: The two cases

In the second case of Table 2, we have a jump of momenta if

$$
p_A(t_0^-)dq^A{}_{|q(t_0)} \notin (D^+_{q(t_0)})^\perp .
$$

$$\boxed{\begin{cases} (p_A(t_0^+) - p_A(t_0^-))\, dq^A{}_{|q(t_0)} \in D^+_{q(t_0)} \\ p_A(t_0^+)\, dq^A{}_{|q(t_0)} \in (D^+_{q(t_0)})^\perp \end{cases}}$$

Table 3: Jump equations

In such a case, our proposal for the equations which determine the jump is contained in Table 3.

An explicit derivation of the momentum jumps for Cases 1 and 3 would be as follows. Let m be the maximum between $\rho = r$, the rank at the regular preceding points, and $\rho = s$, the rank at the regular posterior points. Then there exists a neighbourhood U of $q(t_0)$ and 1-forms $\omega_1, ..., \omega_m$ such that

$$D_q = \operatorname{span}\{\omega_1(q), ..., \omega_m(q)\}\ ,\ \forall q \in U\,.$$

Let us suppose that $\omega_1, ..., \omega_s$ are linearly independent at the regular posterior points (if not, we reorder them). Obviously, at $q(t_0)$, these s 1-forms are linearly dependent. In the following, we will denote $\omega_i \equiv \omega_i(q(t))$ (t time inmediately posterior to t_0), in order to simplify notation.

Since the lagrangian is of the form $L = \frac{1}{2}g - U = \frac{1}{2}g_{AB}\dot{q}^A\dot{q}^B - U(q)$, then we have that

$$\omega_{jA}(q(t))\dot{q}^A(t) = \sum_{A,B} \omega_{jA} g^{AB} p_B(t) = 0\,,\ j = 1, ..., s\,. \tag{8}$$

Using the metric g we have the decomposition $T^*_qQ = D_q \oplus D_q^\perp$, $q \in Q$. The two complementary projectors associated to this decomposition are

$$\mathcal{P}_q : T^*_qQ \longrightarrow D_q^\perp\,,\ \ \mathcal{Q}_q : T^*_qQ \longrightarrow D_q\,.$$

The projector $\mathcal{P}_q$ is given by $\mathcal{P}_q(\alpha_q) = \alpha_q - \mathcal{C}^{ij}\alpha_q(Z_i)\omega_j$, $\alpha_q \in T^*Q$, where

$$Z_i = g^{AB}\omega_{iB}\frac{\partial}{\partial q^A}|_q\,,$$

and $\mathcal{C}^{ij}$ are the entries of the inverse matrix of $\mathcal{C}$, the symmetric matrix with entries $\mathcal{C}_{ij} = \omega_{iA}g^{AB}\omega_{jB}$.

By definition $p_A(t_0)_+ dq^A|_{q(t_0)} = \lim_{t\to t_0^+}(p_A(t)dq^A|_{q(t)})$. From (8), we have $\mathcal{P}_{q(t)}(p_A(t)dq^A|_{q(t)})) = p_A(t)dq^A|_{q(t)}$ and then

$$p_A(t_0)_+ dq^A|_{q(t_0)} = \left(\lim_{t\to t_0^+}\mathcal{P}_{q(t)}\right)(p_A(t_0)_+ dq^A|_{q(t_0)}) \in (D^+_{q(t_0)})^\perp\,. \tag{9}$$

Combining (9) with the other equation in Table 3, we get

$$p_A(t_0)_+ dq^A|_{q(t_0)} = \left(\lim_{t\to t_0^+} \mathcal{P}_{q(t)}\right)\left[p_A(t_0)dq^A|_{q(t_0)}\right] .$$

In coordinates, this can be expressed as

$$p_A(t_0)_+ = p_A(t_0) - \lim_{t\to t_0^+}\left(\sum_{i,j,A,B} \mathcal{C}^{ij}\omega_{jB}g^{BC}\omega_{iA}\right)\Big|_{q(t)} p_C(t_0)\,,\ A = 1,...,n\,. \tag{10}$$

6 The rolling sphere

Consider a homogeneous sphere rolling on a plane. The configuration space is $Q = \mathbb{R}^2 \times SO(3)$: (x, y) denotes the position of the center of the sphere and (φ, θ, ψ) denote the eulerian angles.

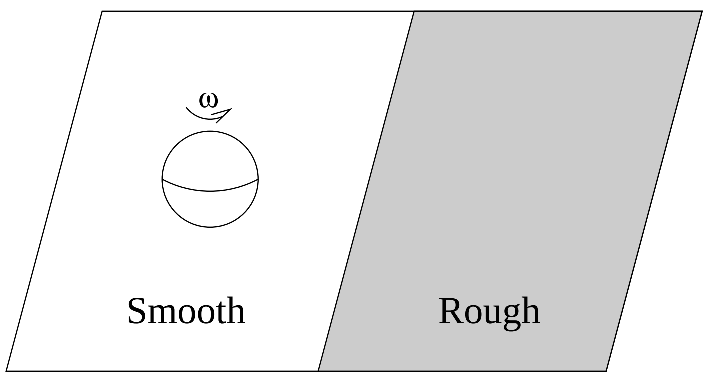

Figure 1: The rolling sphere on a " special" surface

Let us suppose that the plane is smooth if $x < 0$ and absolutely rough if $x > 0$ (see Figure 1). On the smooth part, we assume that the motion of the ball is free, that is, the sphere can slip. But if it reaches the rough half-plane, the sphere begins rolling without slipping, because of the presence of the constraints imposed by the roughness. We are interested in knowing the trajectories of the sphere and, in particular, the possible changes in its dynamics because of the crossing from one half-plane to the other.

The kinetic energy of the sphere is

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\right) ,$$

where ω_x, ω_y and ω_z are the angular velocities given by

$$\begin{aligned} \omega_x &= \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi\,, \\ \omega_y &= \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi\,, \\ \omega_z &= \dot{\varphi}\cos\theta + \dot{\psi}\,. \end{aligned}$$

The potential energy is not considered here since it is constant.

The condition of rolling without sliding of the sphere when $x > 0$ implies that the point of contact of the sphere and the plane has zero velocity

$$\begin{aligned} \phi^1 &= \dot{x} - r\omega_y = 0\,, \\ \phi^2 &= \dot{y} + r\omega_x = 0\,, \end{aligned}$$

where r is the radius of the sphere. Following the classical procedure [21], we introduce quasi-coordinates "q_1", "q_2" and "q_3" such that "$\dot{q}_1$"$= \omega_x$, "$\dot{q}_2$"$= \omega_y$ and "$\dot{q}_3$"$= \omega_z$. The nonholonomic generalizad differentiable codistribution D is given by

$$D_{(x,y,\phi,\theta,\psi)} = \begin{cases} \{0\}\,, & \text{if } x \leq 0\,, \\ \operatorname{span}\{dx - rdq^2, dy + rdq^1\}\,, & \text{if } x > 0\,. \end{cases}$$

The intersection of the regular set of the generalized codistribution and the (x, y)-plane has two connected components, the half-planes $R_1 = \{x < 0\}$ and $R_2 = \{x > 0\}$. The line $\{x = 0\}$ belongs to the singular set of D.

On R_1 the codistribution is zero, so the motion equations are

$$\begin{aligned} m\ddot{x} &= 0\,, & mk^2\dot{\omega}_x &= 0\,, \\ m\ddot{y} &= 0\,, & mk^2\dot{\omega}_y &= 0\,, \\ & & mk^2\dot{\omega}_z &= 0\,. \end{aligned} \tag{11}$$

On R_2 we have to take into account the constraints to obtain the following equations of motion

$$\begin{aligned} m\ddot{x} &= \lambda_1\,, & mk^2\dot{\omega}_x &= r\lambda_2\,, \\ m\ddot{y} &= \lambda_2\,, & mk^2\dot{\omega}_y &= -r\lambda_1\,, \\ & & mk^2\dot{\omega}_z &= 0\,, \end{aligned} \tag{12}$$

with the constraint equations $\dot{x} - r\omega_y = 0$ and $\dot{y} + r\omega_x = 0$. One can compute the Lagrange multipliers by an algebraic procedure (see [16]).

Suppose that the sphere starts its motion at a point of R_1 with the following initial conditions at time $t = 0$: $x_0 < 0$, y_0, $\dot{x}_0 > 0$, $\dot{y}_0$, $(\omega_x)_0$, $(\omega_y)_0$ and $(\omega_z)_0$. Integrating equations (11) we have that if $x(t) < 0$

$$\begin{aligned} x(t) &= \dot{x}_0 t + x_0\,, & \omega_x(t) &= (\omega_x)_0\,, \\ y(t) &= \dot{y}_0 t + y_0\,, & \omega_y(t) &= (\omega_y)_0\,, \\ & & \omega_z(t) &= (\omega_z)_0\,. \end{aligned} \tag{13}$$

At time $\bar{t} = -x_0/\dot{x}_0$ the sphere finds the rough surface of the plane, where the codistribution is no longer zero and it is suddenly forced to roll without sliding (Case 1). Following the discussion in Section 5, we calculate the instantaneous change of velocity (momentum) at $x = 0$.

First of all we compute the matrix

$$\mathcal{C} = (1 + r^2 k^{-2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Next, a direct computation shows that the projector $\mathcal{P}$ does not depend on the base point

$$\mathcal{P} = \begin{pmatrix} \frac{r^2}{r^2+k^2} & 0 & 0 & \frac{r}{r^2+k^2} & 0 \\ 0 & \frac{r^2}{r^2+k^2} & -\frac{r}{r^2+k^2} & 0 & 0 \\ 0 & \frac{-rk^2}{r^2+k^2} & \frac{k^2}{r^2+k^2} & 0 & 0 \\ \frac{rk^2}{r^2+k^2} & 0 & 0 & \frac{k^2}{r^2+k^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

Therefore, we have

$$\begin{array}{rclrcl} (p_x)_+ & = & \frac{r^2(p_x)_0+r(p_2)_0}{r^2+k^2} \,, & (p_1)_+ & = & \frac{-rk^2(p_y)_0+k^2(p_1)_0}{r^2+k^2} \,, \\ (p_y)_+ & = & \frac{r^2(p_y)_0-r(p_1)_0}{r^2+k^2} \,, & (p_2)_+ & = & \frac{rk^2(p_x)_0+k^2(p_2)_0}{r^2+k^2} \,, \\ & & & (p_3)_+ & = & (p_3)_0 \,. \end{array}$$

Now, using the relation between the momenta and the quasi-velocities $p_x = \dot{x}$, $p_y = \dot{y}$, $p_1 = k^2\omega_x$, $p_2 = k^2\omega_y$, $p_3 = k^2\omega_z$, we deduce that

$$\begin{array}{rclrcl} \dot{x}_+ & = & \frac{r^2\dot{x}_0+rk^2(\omega_y)_0}{r^2+k^2} \,, & (\omega_x)_+ & = & \frac{-r\dot{y}_0+k^2(\omega_x)_0}{r^2+k^2} \,, \\ \dot{y}_+ & = & \frac{r^2\dot{y}_0-rk^2(\omega_x)_0}{r^2+k^2} \,, & (\omega_y)_+ & = & \frac{r\dot{x}_0+k^2(\omega_y)_0}{r^2+k^2} \,, \\ & & & (\omega_z)_+ & = & (\omega_z)_0 \,. \end{array} \tag{14}$$

Finally, integrating equations (12) at time $\bar{t} = -x_0/\dot{x}_0$ with initial conditions given by (14) we obtain that if $t > \bar{t}$

$$\begin{array}{rclrcl} x(t) & = & \frac{r^2\dot{x}_0+rk^2(\omega_y)_0}{r^2+k^2}(t-\bar{t}) \,, & \omega_x(t) & = & \frac{-r\dot{y}_0+k^2(\omega_x)_0}{r^2+k^2} \,, \\ y(t) & = & \frac{r^2\dot{y}_0-rk^2(\omega_x)_0}{r^2+k^2}(t-\bar{t}) + \dot{y}_0\bar{t} + y_0 \,, & \omega_y(t) & = & \frac{r\dot{x}_0+k^2(\omega_y)_0}{r^2+k^2} \,, \\ & & & \omega_z(t) & = & (\omega_z)_0 \,. \end{array} \tag{15}$$

Acknowledgments

This work was partially supported by grant DGICYT (Spain) PB97-1257. J. Cortés and S. Martínez wish to thank Spanish Ministerio de Educación y

Cultura for FPU and FPI grants, respectively. The authors wish to thank F. Cantrijn for helpful comments and suggestions.

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