A catalog of inverse-kinematics planners for underactuated systems on matrix Lie groups

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Abstract— **This paper presents motion planning algorithms for underactuated systems evolving on rigid rotation and displacement groups. Motion planning is transcribed into (low-dimensional) combinatorial selection and inverse-kinematics problems. We present a catalog of solutions for all underactuated systems on $\mathrm{SE}(2)$, $\mathrm{SO}(3)$ and $\mathrm{SE}(2) \times \mathbb{R}$ classified according to their controllability properties.**

I. Introduction

This paper presents motion planning algorithms for kinematic models of underactuated mechanical systems; we consider kinematic (i.e., driftless) models that are switched control systems, that is, dynamical systems described by a family of admissible vector fields and a control strategy that governs the switching between them. In particular, we focus on families of left-invariant vector fields defined on rigid displacements subgroups.

This class of systems arises in the context of kinematic modeling and kinematic reductions for mechanical control systems; see the recent works [1], [2], [3], [4], [5]. Clearly, the transcription into kinematic models simplifies the motion planning problem; e.g., [4] discusses 3R planar manipulators and [6], [7] discuss the snakeboard system.

Literature review

Motion planning for kinematic models, sometimes referred to as driftless or nonholonomic models, is a classic problem in robotics; see [8] and also the references therein. In particular, the algorithms in [9], [10], [11] focus on dynamical aspects and exploit controllability properties.

For the class of systems of interest in this paper, the search for a motion planning algorithm is closely related to the inverse-kinematics problem. Example inverse-kinematics methods include (i) iterative numerical methods for nonlinear optimization, see [12], (ii) geometric and decoupling methods for classes of manipulators, see [13], [14], (iii) the Paden-Kahan subproblems approach, see [15], [11], and (iv) the general polynomial programming approach, see [16]. The latter and more general method is based on tools from algebraic geometry and relies on simultaneously solving systems of algebraic equations. Despite these efforts, no general methodology is currently available to solve these problems in closed-form. Accordingly, it is common to provide and catalog closed-form solutions for classes of relevant example systems; see [11], [13], [14].

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Problem statement

We consider left-invariant control systems evolving on a matrix Lie subgroup $\mathrm{G} \subset \mathrm{SE}(3)$. Examples include systems on $\mathrm{SE}(2)$, $\mathrm{SO}(3)$ and $\mathrm{SE}(2) \times \mathbb{R}$. As usual in Lie group theory, we identify left-invariant vector fields with their value at the identity. Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G, consider the associated driftless control system

$$\dot{g}(t) = \sum_{i=1}^{m} V_i(g(t))u_i(t)\,, \tag{1}$$

where $g : \mathbb{R} \to \mathrm{G}$ and where the controls $(u_1, \dots, u_m)$ take value in $\{(\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\}$. For these systems, controllability can be assessed by algebraic means: it suffices to check the lack of involutivity of $\mathrm{span}\{V_1, \dots, V_m\}$. Recall that for matrix Lie algebras, Lie brackets are matrix commutators $[A, B] = AB - BA$.

This paper addresses the problem of how to compute feasible motion plans for the control system (1) by concatenating a finite number of flows along the input vector fields. We call a flow along any input vector field a *motion primitive* and its duration a *coasting time*. Therefore, motion planning is reduced to the problem of selecting a finite-length combination of k motion primitives $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ and computing appropriate coasting times $(t_1, \dots, t_k) \in \mathbb{R}^k$ that steer the system from the identity in the group to any target configuration $g_{\mathrm{f}} \in \mathrm{G}$. In mathematical terms, we need to solve

$$g_{\mathrm{f}} = \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k}).$$

Hence, motion planning is transcribed into low-dimensional combinatorial selection and inverse-kinematics problems.

Contribution

The contribution of this paper is a catalog of solutions for underactuated example systems defined on $\mathrm{SE}(2)$, $\mathrm{SO}(3)$, or $\mathrm{SE}(2) \times \mathbb{R}$. Based on a controllability analysis, we classify families of underactuated systems that pose qualitatively different planning problems. For each family, we solve the planning problem by providing *a* combination of k motion primitives and corresponding closed-form expressions for the coasting times. In each case, we attempt to select $k = \dim(\mathrm{G})$: generically, this is the minimum necessary (but sometimes not sufficient) number of motion primitives needed. If the motion planning algorithm entails exactly dim(G) motion primitives, i.e., minimizes the number of switches, we will refer to it as a *switch-optimal* algorithm. Sections II, III, and IV present switch-optimal planners for SE(2), SO(3), and SE(2) $\times\ \mathbb{R}$, respectively. Due to reasons of space, we refer to the journal version [17] of this paper for the proof of all results concerning SE(2) $\times\ \mathbb{R}$.

Notation

Here we briefly collect the notation used throughout the paper. Let S be a set, $\mathrm{id}_S\colon S \to S$ denote the identity map on S and let $\mathrm{ind}_S\colon \mathbb{R} \to \mathbb{R}$ denote the characteristic function of S, i.e., $\mathrm{ind}_S(x) = 1$ if $x \in S$ and $\mathrm{ind}_S(x) = 0$ if $x \notin S$. Let $\arctan2\,(x, y)$ denote the arctangent of y/x taking into account which quadrant the point (x, y) is in. We make the convention $\arctan2\,(0, 0) = 0$. Let $\mathrm{sign}\colon \mathbb{R} \to \mathbb{R}$ be the sign function, i.e., $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(0) = 0$. Let A_{ij} be the (i, j) element of the matrix A. Given $v, w \in \mathbb{R}^n$, let $\arg(v, w) \in [0, \pi]$ denote the angle between them. Let $\|\cdot\|$ denote the Euclidean norm.

Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G, we associate to each multiindex $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ the forward-kinematics map $\mathcal{FK}^{(i_1,\dots,i_k)}\colon \mathbb{R}^k \to \mathrm{G}$ given by $(t_1, \dots, t_k) \mapsto \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k})$.

II. Catalog for SE(2)

Let $\{e_\theta, e_x, e_y\}$ be the basis of $\mathfrak{se}(2)$:

$$e_\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $[e_\theta, e_x] = e_y$, $[e_y, e_\theta] = e_x$ and $[e_x, e_y] = 0$. For ease of presentation, we write $V \in \mathfrak{se}(2)$ as $V = a e_\theta + b e_x + c e_y \equiv (a, b, c)$, and $g \in \mathrm{SE}(2)$ as

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \equiv (\theta, x, y)\,.$$

With this notation, $\exp\colon \mathfrak{se}(2) \to \mathrm{SE}(2)$ is

$$\exp(a, b, c) = \left(a\,,\ \frac{\sin a}{a} b - \frac{1-\cos a}{a} c\,,\ \frac{1-\cos a}{a} b + \frac{\sin a}{a} c\right)$$

for $a \neq 0$, and $\exp(0, b, c) = (0, b, c)$.

Lemma II.1: (Controllability conditions). Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{se}(2)$. Their Lie closure is full rank if and only if $a_1 b_2 - b_1 a_2 \neq 0$ or $c_1 a_2 - a_1 c_2 \neq 0$.

Proof: Given $[V_1, V_2] = (0, c_1 a_2 - a_1 c_2\,,\ a_1 b_2 - b_1 a_2)$, one can see that $\mathrm{span}\,\{V_1, V_2, [V_1, V_2]\} = \mathfrak{se}(2)$ if and only if $(a_1 b_2 - b_1 a_2)^2 + (c_1 a_2 - a_1 c_2)^2 \neq 0$. ■

Let $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ satisfy the controllability condition in Lemma II.1. Accordingly, either a_1 or a_2 is different from zero. Without loss of generality, we will assume that $a_1 \neq 0$, and take $a_1 = 1$. As a consequence of Lemma II.1, there are two qualitatively different cases to be considered:

$$\mathcal{S}_1 = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1),\ V_2 = (0, b_2, c_2) \text{ and } b_2^2 + c_2^2 = 1\}\,,$$
$$\mathcal{S}_2 = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1),\ V_2 = (1, b_2, c_2) \text{ and either } b_1 \neq b_2 \text{ or } c_1 \neq c_2\}\,.$$

Since $\dim(\mathfrak{se}(2)) = 3$, we need at least three motion primitives along the flows of $\{V_1, V_2\}$ to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1,2,1)}\colon \mathbb{R}^3 \to \mathrm{SE}(2)$. In the following propositions, we compute solutions for $\mathcal{S}_1$ and $\mathcal{S}_2$-systems.

Proposition II.2: (Inversion for $\mathcal{S}_1$-systems on SE(2)). Let $(V_1, V_2) \in \mathcal{S}_1$. Consider the map $\mathcal{IK}[\mathcal{S}_1]\colon \mathrm{SE}(2) \to \mathbb{R}^3$,

$$\mathcal{IK}[\mathcal{S}_1](\theta, x, y) = (\arctan2\,(\alpha, \beta)\,, \rho, \theta - \arctan2\,(\alpha, \beta)),$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{S}_1]$ is a global right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK}[\mathcal{S}_1] = \mathrm{id}_{\mathrm{SE}(2)}\colon \mathrm{SE}(2) \to \mathrm{SE}(2)$. Note that the algorithm provided in the proposition is not only switch-optimal, but also works globally.

Proof: The proof follows from the expression of the map $\mathcal{FK}^{(1,2,1)}$. Let $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = (\theta, x, y)$,

$$\theta = t_1 + t_3\,,$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2\,.$$

The equation in $[x, y]^T$ can be rewritten as $[\alpha, \beta]^T = [\cos t_1, \sin t_1]^T t_2$. The selection $t_1 = \arctan2\,(\alpha, \beta)$, $t_2 = \rho$ solves this equation. ■

Proposition II.3: (Inversion for $\mathcal{S}_2$-systems on SE(2)). Let $(V_1, V_2) \in \mathcal{S}_2$. Define the neighborhood of the identity in SE(2)

$$U = \{(\theta, x, y) \in \mathrm{SE}(2) \mid \ \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2\,, 2(1-\cos\theta)\|(b_1, c_1)\|^2\}.$$

Consider the map $\mathcal{IK}[\mathcal{S}_2]\colon U \subset \mathrm{SE}(2) \to \mathbb{R}^3$ whose components are

$$\mathcal{IK}[\mathcal{S}_2]_1(\theta, x, y) = \arctan2\left(\rho, \sqrt{4-\rho^2}\right) + \arctan2\,(\alpha, \beta)\ ,$$
$$\mathcal{IK}[\mathcal{S}_2]_2(\theta, x, y) = \arctan2\left(2-\rho^2, \rho\sqrt{4-\rho^2}\right),$$
$$\mathcal{IK}[\mathcal{S}_2]_3(\theta, x, y) = \theta - \mathcal{IK}[\mathcal{S}_2]_1(\theta, x, y) - \mathcal{IK}[\mathcal{S}_2]_2(\theta, x, y)\,,$$

and $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|^2} \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{S}_2]$ is a local right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK}[\mathcal{S}_2] = \mathrm{id}_U : U \to U$.

Proof: If $(\theta, x, y) \in U$, then

$$\rho = \|(\alpha,\beta)\| \leq \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|} \cdot \left(\|(x,y)\| + \left\| \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} \right\| \right) \leq 2\,,$$

and hence $\mathcal{IK}[\mathcal{S}_2]$ is well-defined on U. Let $\mathcal{IK}[\mathcal{S}_2](\theta,x,y) = (t_1,t_2,t_3)$. The components of $\mathcal{FK}^{(1,2,1)}(t_1,t_2,t_3)$ are

$$\begin{aligned} &\mathcal{FK}_1^{(1,2,1)}(t_1,t_2,t_3) = t_1 + t_2 + t_3\,, \\ &\begin{bmatrix} \mathcal{FK}_2^{(1,2,1)}(t_1,t_2,t_3) \\ \mathcal{FK}_3^{(1,2,1)}(t_1,t_2,t_3) \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} \\ &\qquad + \begin{bmatrix} c_1 - c_2 & b_1 - b_2 \\ b_2 - b_1 & c_1 - c_2 \end{bmatrix} \begin{bmatrix} \cos t_1 - \cos(t_1+t_2) \\ \sin t_1 - \sin(t_1+t_2) \end{bmatrix}. \end{aligned}$$

In an analogous way to the previous proof, one verifies $\mathcal{FK}^{(1,2,1)}(t_1,t_2,t_3) = (\theta,x,y)$. ■

Remark II.4: The map $\mathcal{IK}[\mathcal{S}_2]$ in Proposition II.3 is a local right inverse to $\mathcal{FK}^{(1,2,1)}$ on a domain that strictly contains U. In other words, our estimate of the domain of $\mathcal{IK}[\mathcal{S}_2]$ is conservative. For instance, for points of the form $(0,x,y) \in \mathrm{SE}(2)$, it suffices to ask for

$$\|(x,y)\| \leq 2\|(c_1 - c_2, b_1 - b_2)\|\,.$$

For a point $(\theta,0,0) \in \mathrm{SE}(2)$, it suffices to ask for

$$(1-\cos\theta)\|(b_1,c_1)\|^2 \leq 2\|(c_1-c_2, b_1-b_2)\|^2\,.$$

Additionally, without loss of generality, it is convenient to assume that the vector fields V_1, V_2 satisfy $b_1^2 + c_1^2 \leq b_2^2 + c_2^2$, so as to maximize the domain U.

We illustrate the performance of the algorithms in Fig. 1.

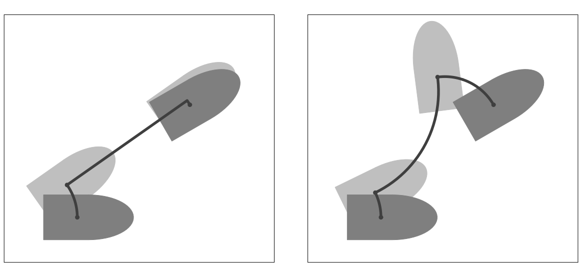

Fig. 1. We illustrate the inverse-kinematics planners for $\mathcal{S}_1$ and $\mathcal{S}_2$-systems. The parameters of both systems are $(b_1,c_1) = (0,.5)$, $(b_2,c_2) = (1,0)$. The target location is $(\pi/6,1,1)$. Initial and target locations are depicted in dark gray.

III. Catalog for SO(3)

Let $\{\widehat{e}_x, \widehat{e}_y, \widehat{e}_z\}$ be the basis of $\mathfrak{so}(3)$:

$$\widehat{e}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \widehat{e}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \widehat{e}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we make use of the notation $\widehat{V} = a\widehat{e}_x + b\widehat{e}_y + c\widehat{e}_z \equiv \widehat{(a,b,c)}$ based on the Lie algebra isomorphism $\widehat{\cdot}$: $(\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\cdot,\cdot])$. Rodrigues formula [11] for the exponential $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is

$$\exp(\widehat{\eta}) = I_3 + \frac{\sin\|\eta\|}{\|\eta\|}\widehat{\eta} + \frac{1-\cos\|\eta\|}{\|\eta\|^2}\widehat{\eta}^2\,.$$

The commutator relations are $[\widehat{e}_x, \widehat{e}_z] = -\widehat{e}_y$, $[\widehat{e}_y, \widehat{e}_z] = \widehat{e}_x$ and $[\widehat{e}_x, \widehat{e}_y] = \widehat{e}_z$.

Lemma III.1: (Controllability conditions). Consider two left-invariant vector fields $V_1 = (a_1,b_1,c_1)$ and $V_2 = (a_2,b_2,c_2)$ in $\mathfrak{so}(3)$. Their Lie closure is full rank if and only if $c_1a_2 - a_1c_2 \neq 0$ or $b_1c_2 - c_1b_2 \neq 0$ or $b_1a_2 - a_1b_2 \neq 0$.

Proof: Given the equality $[\widehat{V}_1, \widehat{V}_2] = \widehat{V_1 \times V_2}$, with $V_1 \times V_2 = (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$, one can see that $\mathrm{span}\,\{V_1, V_2, [V_1,V_2]\} = \mathfrak{so}(3)$ if and only if

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ b_1c_2 - b_2c_1 & c_1a_2 - c_2a_1 & a_1b_2 - a_2b_1 \end{bmatrix} = (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2 \neq 0\,.$$

■

Let V_1, V_2 satisfy the controllability condition in Lemma III.1. Without loss of generality, we can assume $V_1 = e_z$ (otherwise we perform a suitable change of coordinates), and $\|V_2\| = 1$. In what follows, we let $V_2 = (a,b,c)$. Since e_z and V_2 are linearly independent, necessarily $a^2 + b^2 \neq 0$ and $c \neq \pm 1$. Since $\dim(\mathfrak{so}(3)) = 3$, we need at least three motion primitives to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1,2,1)} : \mathbb{R}^3 \to \mathrm{SO}(3)$, that is

$$\mathcal{FK}^{(1,2,1)}(t_1,t_2,t_3) = \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)\exp(t_3\widehat{e}_z)\,. \quad (2)$$

Observe that equation (2) is similar to the formula for certain sets of Euler angles; see [11].

Proposition III.2: (Inversion for systems on SO(3)). Let $V_1 = (0,0,1)$ and $V_2 = (a,b,c)$, with $a^2 + b^2 \neq 0$ and $c \neq \pm 1$. Define the neighborhood of the identity in SO(3)

$$U = \{R \in \mathrm{SO}(3) \mid R_{33} \in [2c^2 - 1, 1]\}.$$

Consider the map $\mathcal{IK} : U \subset \mathrm{SO}(3) \to \mathbb{R}^3$ whose components are

$$\begin{aligned} \mathcal{IK}_1(R) &= \mathrm{arctan2}\left(w_1R_{13} + w_2R_{23}, -w_2R_{13} + w_1R_{23}\right)\,, \\ \mathcal{IK}_2(R) &= \arccos\left(\frac{R_{33} - c^2}{1 - c^2}\right)\,, \\ \mathcal{IK}_3(R) &= \mathrm{arctan2}\left(v_1R_{31} + v_2R_{32}, v_2R_{31} - v_1R_{32}\right)\,, \end{aligned}$$

where, for $z = (1 - \cos(\mathcal{IK}_2(R)), \sin(\mathcal{IK}_2(R)))^T$,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} ac & b \\ cb & -a \end{bmatrix} z\,, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ac & -b \\ cb & a \end{bmatrix} z.$$

Then, $\mathcal{IK}$ is a local right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK} = \mathrm{id}_U : U \to U$.

Proof: Let $R \in U$. Then, $|\frac{R_{33}-c^2}{1-c^2}| \leq 1$, and hence $\mathcal{IK}(R)$ is well-defined. Denote $t_i = \mathcal{IK}_i(R)$ and let us show $R = \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$. Recall that the rows (resp. the columns) of a rotation matrix consist of orthonormal vectors in $\mathbb{R}^3$. Therefore, the matrix $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) \in \mathrm{SO}(3)$ is determined by its third column $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)e_z$ and its third row $e_z^T \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$. The factors in (2) admit the following closed-form expressions. For $c_t = \cos t$ and $s_t = \sin t$,

$$\exp(t\widehat{e}_z) = \begin{bmatrix} c_t & -s_t & 0 \\ s_t & c_t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $\exp(t\widehat{V}_2)$ equals

$$\begin{bmatrix} a^2 + (1-a^2)c_t & ba(1-c_t) - cs_t & ca(1-c_t) + bs_t \\ ab(1-c_t) + cs_t & b^2 + (1-b^2)c_t & cb(1-c_t) - as_t \\ ac(1-c_t) - bs_t & bc(1-c_t) + as_t & c^2 + (1-c^2)c_t \end{bmatrix}.$$

Now, using the fact that $\exp(t\widehat{e}_z)e_z = e_z$, we get

$$\begin{aligned} \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)e_z &= \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)\exp(t_3\widehat{e}_z)e_z \\ &= \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)e_z = \exp(t_1\widehat{e}_z)\begin{bmatrix} w_1 \\ w_2 \\ R_{33} \end{bmatrix} = Re_z\,. \end{aligned}$$

A similar computation shows that $e_z^T \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = e_z^T R$, which concludes the proof. ■

Remark III.3: If $\widehat{e}_z$ and V_2 are perpendicular, then $U = SO(3)$ and the map $\mathcal{IK}$ is a global right inverse of $\mathcal{FK}^{(1,2,1)}$. Otherwise, let us provide an equivalent formulation of the constraint $R_{33} \in [2c^2 - 1, 1]$ in terms of the axis/angle representation of the rotation matrix R. Recall that there always exist a, possibly non-unique, rotation angle $\theta \in [0, \pi]$ and an unit-length axis of rotation $\omega \in \mathbb{S}^2$ such that $R = \exp(\widehat{\omega}\theta)$. Because $\widehat{\omega}^2 = \omega^T\omega - I_3$, an equivalent statement of Rodrigues formula is

$$R = I_3 + \widehat{\omega}\sin\theta + (1-\cos\theta)(\omega^T\omega - I_3).$$

From $e_z^T\omega = \cos(\arg(e_z, \omega))$, we compute

$$\begin{aligned} e_z^T Re_z &= e_z^T e_z + (1-\cos\theta)((e_z^T\omega)^2 - e_z^T e_z) \\ &= 1 + (1-\cos\theta)((e_z^T\omega)^2 - 1) \\ &= 1 - \sin^2(\arg(e_z, \omega))(1-\cos\theta)\,. \end{aligned} \quad (3)$$

Therefore, $R_{33} \in [2c^2 - 1, 1]$ if and only if

$$\begin{aligned} &1 - \sin^2(\arg(e_z, \omega))(1-\cos\theta) \geq 2c^2 - 1 \\ &\iff \sin^2(\arg(e_z, \omega))(1-\cos\theta) \leq 2(1-c^2)\,. \end{aligned}$$

Two sufficient conditions are also meaningful. In terms of the rotation angle, if $|\theta| \leq \arccos(2c^2 - 1)$ then $1 - \cos\theta \leq 2(1-c^2)$, and in turn equation (3) is satisfied. In terms of the axis of rotation, a sufficient condition for equation (3) is $\sin^2(\arg(e_z, \omega)) \leq \sin^2(\arg(e_z, V_2)) = 1 - c^2$.

We illustrate the performance of the algorithm in Fig. 2.

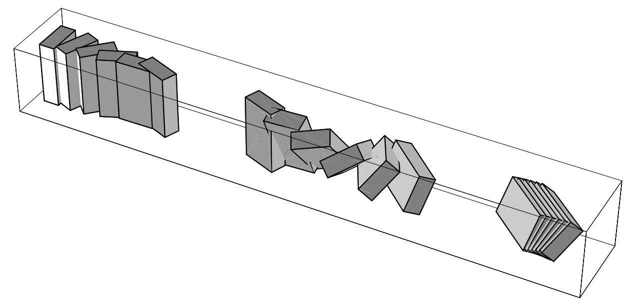

Fig. 2. We illustrate the inverse-kinematics planner on SO(3). The system parameters are $(a, b, c) = (0, 1/\sqrt{2}, 1/\sqrt{2})$. The target final rotation is $\exp(\pi/3, \pi/3, 0)$. To render the sequence of three rotations visible, the body is translated along the inertial x-axis.

IV. Catalog for SE(2) × ℝ

Let $\{(e_\theta, 0), (e_x, 0), (e_y, 0), (0, 0, 0, 1)\}$ be a basis of $\mathfrak{se}(2) \times \mathbb{R}$, where $\{e_\theta, e_x, e_y,\}$ stands for the basis of $\mathfrak{se}(2)$ introduced in Section II. With a slight abuse of notation, we let e_θ denote $(e_\theta, 0)$, and we similarly redefine e_x and e_y. We also let $e_z = (0, 0, 0, 1)$. The only non-vanishing Lie algebra commutators are $[e_\theta, e_x] = e_y$ and $[e_\theta, e_y] = -e_x$.

A left-invariant vector field V in $\mathfrak{se}(2) \times \mathbb{R}$ is written as $V = ae_\theta + be_x + ce_y + de_z \equiv (a, b, c, d)$, and $g \in \mathrm{SE}(2) \times \mathbb{R}$ as $g = (\theta, x, y, z)$. The exponential map, $\exp : \mathfrak{se}(2) \times \mathbb{R} \longrightarrow \mathrm{SE}(2) \times \mathbb{R}$, is given component-wise by the exponential on $\mathfrak{se}(2)$ and $\mathbb{R}$, respectively. That is, $\exp(V)$ is equal to

$$\left(a\,,\ \frac{\sin a}{a}b - \frac{1-\cos a}{a}c\,,\ \frac{1-\cos a}{a}b + \frac{\sin a}{a}c\,,\ d\right)$$

if $a \neq 0$, and $\exp(V) = (0, b, c, d)$ if $a = 0$.

Lemma IV.1: (Controllability conditions for systems in $\mathrm{SE}(2) \times \mathbb{R}$ with 2 inputs). Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1, d_1)$ and $V_2 = (a_2, b_2, c_2, d_2)$ in $\mathfrak{se}(2) \times \mathbb{R}$. Their Lie closure is full rank if and only if $a_2d_1 - d_2a_1 \neq 0$, and either $c_1a_2 - a_1c_2 \neq 0$ or $a_1b_2 - b_1a_2 \neq 0$.

Let V_1, V_2 satisfy the controllability condition in Lemma IV.1. Without loss of generality, we can assume $a_1 = 1$. As in the case of SE(2), there are two qualitatively different situations to be considered:

$$\begin{aligned} \mathcal{T}_1 = \{(V_1, V_2) \in (\mathfrak{se}(2) \times \mathbb{R})^2 \mid V_1 = (1, b_1, c_1, d_1), \\ V_2 = (0, b_2, c_2, 1) \text{ and } b_2^2 + c_2^2 \neq 0\}\,, \\ \mathcal{T}_2 = \{(V_1, V_2) \in (\mathfrak{se}(2) \times \mathbb{R})^2 \mid V_1 = (1, b_1, c_1, d_1), \\ V_2 = (1, b_2, c_2, d_2), d_1 \neq d_2 \text{ and either } b_1 \neq b_2 \text{ or } c_1 \neq c_2\}\,. \end{aligned}$$

Lemma IV.2: (Controllability conditions for $\mathrm{SE}(2) \times \mathbb{R}$ systems with 3 inputs). Consider three left-invariant vector fields $V_i = (a_i, b_i, c_i, d_i)$, $i = 1, 2, 3$ in $\mathfrak{se}(2) \times \mathbb{R}$. Assume $\overline{\mathrm{Lie}}(\{V_{i_1}, V_{i_2}\}) \subsetneq \mathfrak{se}(2) \times \mathbb{R}$, for $i_j \in \{1, 2, 3\}$ and $\overline{\mathrm{Lie}}(\{V_1, V_2, V_3\}) = \mathfrak{se}(2) \times \mathbb{R}$. Then, possibly after a reordering of the vector fields, they must fall in one of the following cases:

$$\begin{aligned} \mathcal{T}_3 = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1), V_2 = \\ (0, b_2, c_2, 0), V_3 = (1, b_1, c_1, d_3), d_1 \neq d_3 \text{ and } b_2^2 + c_2^2 \neq 0\}, \end{aligned}$$

$$\mathcal{T}_4 = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1), V_2 = (0, b_2, c_2, 0), V_3 = (0,0,0,d_3), 0 \neq d_3 \neq d_1 \text{ and } b_2^2 + c_2^2 \neq 0\},$$
$$\mathcal{T}_5 = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1), V_2 = (1, b_2, c_2, d_1), V_3 = (0,0,0,d_3), d_3 \neq 0 \text{ and either } b_2 \neq b_1 \text{ or } c_1 \neq c_2\}.$$

A. Two-dimensional input distribution

Let V_1, V_2 satisfy the controllability condition in Lemma IV.1. Since $\dim(\mathfrak{se}(2) \times \mathbb{R}) = 4$, we need at least four motion primitives to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(2,1,2,1)} \colon \mathbb{R}^4 \to \mathrm{SE}(2) \times \mathbb{R}$.

Proposition IV.3: (Lack of switch-optimal inversion for $\mathcal{T}_1$-systems on $\mathrm{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_1$. Then, the map $\mathcal{FK}^{(2,1,2,1)}$ in not invertible at any neighborhood of the origin.

Remark IV.4: An identical negative result holds if we start taking motion primitives along the flow of V_1 instead of V_2, i.e., if we consider the map $\mathcal{FK}^{(1,2,1,2)} \colon \mathbb{R}^4 \to \mathrm{SE}(2) \times \mathbb{R}$.

Consider the map $\mathcal{FK}^{(1,2,1,2,1)} \colon \mathbb{R}^5 \to \mathrm{SE}(2) \times \mathbb{R}$.

Proposition IV.5: (Inversion for $\mathcal{T}_1$-systems on $\mathrm{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_1$. Consider the map $\mathcal{IK}[\mathcal{T}_1] \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^5$ whose components are

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_1]_1(\theta,x,y,z) &= \pi \mathrm{ind}_{]-\infty,0[}(\gamma-\rho) + \operatorname{arctan2}(\alpha,\beta) \\
&\quad + \operatorname{arctan2}((\rho+\gamma)/2, 0), \\
\mathcal{IK}[\mathcal{T}_1]_2(\theta,x,y,z) &= (\gamma-\rho)/2, \\
\mathcal{IK}[\mathcal{T}_1]_3(\theta,x,y,z) &= \operatorname{arctan2}((\rho^2-\gamma^2)/4, 0) \\
&\quad + \pi\left(\mathrm{ind}_{]-\infty,0[}(\gamma+\rho) - \mathrm{ind}_{]-\infty,0[}(\gamma-\rho)\right), \\
\mathcal{IK}[\mathcal{T}_1]_4(\theta,x,y,z) &= (\gamma+\rho)/2, \\
\mathcal{IK}[\mathcal{T}_1]_5(\theta,x,y,z) &= \theta - \mathcal{IK}[\mathcal{T}_1]_1(\theta,x,y,z) - \mathcal{IK}[\mathcal{T}_1]_3(\theta,x,y,z),
\end{aligned}$$

where $\rho = \sqrt{\alpha^2+\beta^2}$ and

$$\gamma = z - d_1\theta,$$
$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2+c_2^2}\begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1 \\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta \\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}[\mathcal{T}_1]$ is a global right inverse of $\mathcal{FK}^{(1,2,1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,2,1)} \circ \mathcal{IK}[\mathcal{T}_1] = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}} \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathrm{SE}(2) \times \mathbb{R}$.

Proposition IV.6: (Inversion for $\mathcal{T}_2$-systems on $\mathrm{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_2$. Define the neighborhood of the identity in $\mathrm{SE}(2) \times \mathbb{R}$

$$\begin{aligned}
U = \Big\{ &(\theta,x,y,z) \in \mathrm{SE}(2)\times\mathbb{R} \mid\ 4\|(c_1-c_2, b_1-b_2)\|^2 \geq \\
&\max\{\|(x,y)\|^2, 2(1-\cos\theta)\|(b_1,c_1)\|^2\}, \\
&|z-d_1\theta| \leq 2|d_2-d_1| \arccos\Big(-1 + \frac{1}{\|(c_1-c_2,b_1-b_2)\|} \\
&\cdot \left(\|(x,y)\| + \|(b_1,c_1)\|\sqrt{2(1-\cos\theta)}\right)\Big)\Big\}.
\end{aligned}$$

Consider the map $\mathcal{IK}[\mathcal{T}_2] \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^5$ whose components are

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_2]_1(\theta,x,y,z) &= \operatorname{arctan2}\left(l, \sqrt{4-l^2}\right) + \operatorname{arctan2}(\alpha,\beta), \\
\mathcal{IK}[\mathcal{T}_2]_2(\theta,x,y,z) &= 2\operatorname{arctan2}\left(\sqrt{4-l^2}, l\right), \\
\mathcal{IK}[\mathcal{T}_2]_3(\theta,x,y,z) &= -\operatorname{arctan2}\left(\rho - l, \sqrt{4-(\rho-l)^2}\right) \\
&\quad - \mathcal{IK}[\mathcal{T}_2]_1(\theta,x,y,z) - \mathcal{IK}[\mathcal{T}_2]_2(\theta,x,y,z), \\
\mathcal{IK}[\mathcal{T}_2]_4(\theta,x,y,z) &= \gamma - \mathcal{IK}[\mathcal{T}_2]_2(\theta,x,y,z), \\
\mathcal{IK}[\mathcal{T}_2]_5(\theta,x,y,z) &= \theta - \sum_{i=1}^{4}\mathcal{IK}[\mathcal{T}_2]_i(\theta,x,y,z),
\end{aligned}$$

where $\rho = \sqrt{\alpha^2+\beta^2}$, $s = \sin(\gamma/2)$, $c = \cos(\gamma/2)$ and

$$\gamma = (z - d_1\theta)/(d_2-d_1),$$
$$l = \frac{\rho(1+c) + \mathrm{sign}(\gamma)\sqrt{\rho^2(1+c)^2 - (1+c)(2\rho^2 - 8s^2)}}{2(1+c)},$$
$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\|(d_1-d_2, c_1-c_2)\|^2}\begin{bmatrix} d_1-d_2 & c_2-c_1 \\ c_1-c_2 & d_1-d_2\end{bmatrix} \cdot \left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-d_1 & c_1 \\ c_1 & d_1\end{bmatrix}\begin{bmatrix}1-\cos\theta \\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}[\mathcal{T}_2]$ is a local right inverse of $\mathcal{FK}^{(1,2,1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,2,1)} \circ \mathcal{IK}[\mathcal{T}_2] = \mathrm{id}_U \colon U \to U$.

B. Three-dimensional input distribution

Let V_1, V_2, V_3 satisfy the controllability condition in Lemma IV.2. Consider $\mathcal{FK}^{(1,3,2,1)} \colon \mathbb{R}^4 \to \mathrm{SE}(2) \times \mathbb{R}$.

Proposition IV.7: (Inversion for $\mathcal{T}_3$-systems on $\mathrm{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2, V_3) \in \mathcal{T}_3$. Consider the map $\mathcal{IK}[\mathcal{T}_3] \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^4$ whose components are

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_3]_1(\theta,x,y,z) &= \operatorname{arctan2}(\alpha,\beta) - \mathcal{IK}[\mathcal{T}_3]_2(\theta,x,y,z), \\
\mathcal{IK}[\mathcal{T}_3]_2(\theta,x,y,z) &= \frac{z-d_1\theta}{d_3-d_1}, \\
\mathcal{IK}[\mathcal{T}_3]_3(\theta,x,y,z) &= \rho, \\
\mathcal{IK}[\mathcal{T}_3]_4(\theta,x,y,z) &= \theta - \operatorname{arctan2}(\alpha,\beta),
\end{aligned}$$

where $\rho = \sqrt{\alpha^2+\beta^2}$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2+c_2^2}\begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1 \\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta \\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}[\mathcal{T}_3]$ is a global right inverse of $\mathcal{FK}^{(1,3,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,3,2,1)} \circ \mathcal{IK}[\mathcal{T}_3] = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}} \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathrm{SE}(2) \times \mathbb{R}$.

Consider the map $\mathcal{FK}^{(1,2,1,3)} \colon \mathbb{R}^4 \to \mathrm{SE}(2) \times \mathbb{R}$.

Proposition IV.8: (Inversion for $\mathcal{T}_4$-systems on $\mathrm{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2, V_3) \in \mathcal{T}_4$. Consider the map $\mathcal{IK}[\mathcal{T}_4] \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^4$ given by

$$\mathcal{IK}[\mathcal{T}_4](\theta,x,y,z) = \Big(\operatorname{arctan2}(\alpha,\beta), \rho, \theta - \operatorname{arctan2}(\alpha,\beta), \frac{z-d_1\theta}{d_3}\Big),$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2 + c_2^2}\begin{bmatrix} b_2 & c_2\\ -c_2 & d_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}[\mathcal{T}_4]$ is a global right inverse of $\mathcal{FK}^{(1,2,1,3)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,3)} \circ \mathcal{IK}[\mathcal{T}_4] = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}} \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathrm{SE}(2) \times \mathbb{R}$.

Proposition IV.9: (Inversion for $\mathcal{T}_5$-systems on $\mathrm{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2, V_3) \in \mathcal{T}_5$. Define the neighborhood of the identity in $\mathrm{SE}(2) \times \mathbb{R}$

$$U = \{(\theta, x, y) \in \mathrm{SE}(2) \times \mathbb{R} \mid \ \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x,y)\|^2, 2(1-\cos\theta)\|(b_1, c_1)\|^2\}.$$

Consider the map $\mathcal{IK}[\mathcal{T}_5] \colon U \subset \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^4$ whose components are

$$\mathcal{IK}[\mathcal{T}_5]_1(\theta,x,y,z) = \operatorname{arctan2}\left(\rho, \sqrt{4-\rho^2}\right) + \operatorname{arctan2}(\alpha,\beta)\ ,$$
$$\mathcal{IK}[\mathcal{T}_5]_2(\theta,x,y,z) = \operatorname{arctan2}\left(2-\rho^2, \rho\sqrt{4-\rho^2}\right),$$
$$\mathcal{IK}[\mathcal{T}_5]_3(\theta,x,y,z) = \theta - \mathcal{IK}[\mathcal{T}_5]_1(\theta,x,y) - \mathcal{IK}[\mathcal{T}_5]_2(\theta,x,y)\ ,$$
$$\mathcal{IK}[\mathcal{T}_5]_4(\theta,x,y,z) = \frac{z - d_1\theta}{d_3}\ ,$$

and $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|^2}\begin{bmatrix} c_1 - c_2 & b_2 - b_1\\ b_1 - b_2 & c_1 - c_2\end{bmatrix} \cdot \left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}[\mathcal{T}_5]$ is a local right inverse of $\mathcal{FK}^{(1,2,1,3)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,3)} \circ \mathcal{IK}[\mathcal{T}_5] = \mathrm{id}_U \colon U \to U$.

We illustrate the performance of the algorithms in Fig. 3.

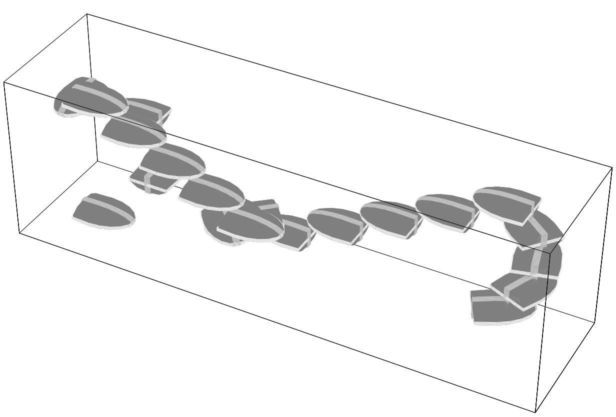

Fig. 3. We illustrate the inverse-kinematics planner for a $\mathcal{T}_1$-system on $\mathrm{SE}(2) \times \mathbb{R}$. The system parameters are $b_1 = 1$, $c_1 = 0$, $d_1 = .5$, $b_2 = -2$, and $c_2 = 0$. The target location is $(\pi/6, 10, 0, 1)$.

V. Conclusions

We have presented a catalog of feasible motion planning algorithms for underactuated controllable systems on SE(2), SO(3) and $\mathrm{SE}(2) \times \mathbb{R}$. Future directions of research include (i) considering other relevant classes of underactuated systems on SE(3), (ii) computing catalogs of optimal sequences of motion primitives, and (iii) developing hybrid feedback schemes that rely on the proposed open-loop planners to achieve point stabilization and trajectory tracking.

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