On synchronous robotic networks – Part II: Time complexity of rendezvous and deployment algorithms

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Abstract

This paper analyzes a number of basic coordination algorithms running on synchronous robotic networks. We provide upper and lower bounds on the time complexity of the move-toward average and circumcenter laws, both achieving rendezvous, and of the centroid law, achieving deployment over a region of interest. The results are derived via novel analysis methods, including a set of results on the convergence rates of linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices.

I. INTRODUCTION

Problem motivation: Although recent years have witnessed the emergence of numerous coordination algorithms for networked mobile systems, the fundamental limits in terms of achievable performance, energy consumption and operational time remain largely unknown. This is partially explained by the inherent difficulty in integrating the various sensing, computing and communication aspects of problems involving groups of mobile agents. In this paper,

Submitted on April 29, 2005

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we consider the problem of analyzing the performance of several coordination algorithms achieving rendezvous and deployment. To this goal, we rely on the general framework proposed in the companion paper [1] to formally model the behavior of robotic networks. Our research effort aims at developing tools and results to assess to what extent coordination algorithms are scalable, and implementable in large networks of mobile agents. Ultimately, we would like to characterize the minimum amount of communication, sensing and control that is necessary to reliably perform a desired task, and we would like to design algorithms that achieve those limits.

Literature review: A survey on cooperative mobile robotics is presented in [2] and an overview of control and communication issues is contained in [3]. Specific topics related to the present treatment include rendezvous [4], [5], [6], [7], [8], cyclic pursuit [9], [10], deployment [11], [12], swarm aggregation [13], gradient climbing [14], flocking [15], [16] and consensus [17], [18], [19]. The papers [20], [21], [10] discuss convergence rates of various motion coordination algorithms. See the aforementioned works for references on additional cooperative strategies designed to perform other spatially-distributed tasks.

Statement of contributions: The companion paper [1] proposes a general framework to model robotic networks and formally analyze their behavior. In particular, this work defines notions of time and communication complexity aimed at capturing the performance and cost of the execution of motion coordination algorithms. Building on these notions, here we establish complexity estimates for various basic motion coordination algorithms that achieve rendezvous and deployment. First, we analyze a simple averaging law for a network of locally-connected agents moving on a line. This law is related to the widely known Vicsek's model, see [15], [22]. We show that this law achieves rendezvous (without preserving connectivity) and that its time complexity belongs to $\Omega(N)$ and $O(N^5)$. Second, for a network of locally-connected agents moving on a line or on a segment, we show that the well-known circumcenter algorithm by [4] has time complexity of order $\Theta(N)$. (This algorithm achieves rendezvous while preserving connectivity with a communication graph with $O(N^2)$ links.) We then consider a network based on a different communication graph, called the limited Delaunay graph, that arises naturally in computational geometry and in the study of wireless communication topologies. For this less dense graph with $O(N)$ communication links, we show that the time complexity of the circumcenter algorithm grows to $\Theta(N^2 \log N)$. For a network of agents moving on $\mathbb{R}^d$ (with a certain communication graph) we introduce a novel "parallel-circumcenter algorithm" and establish its time complexity of order $\Theta(N)$. Third and last, for a network of agents in a one-dimensional environment, we show that the time complexity of the deployment algorithm introduced in [11] is $O(N^3 \log N)$. To obtain these complexity estimates, we develop some novel analysis methods. In particular, we develop a key set of results on linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices that characterize their convergence rates as a function of the matrices dimensions.

Organization: Section II briefly reviews the general approach to the modeling of robotic networks proposed in [1], presenting the notions of control and communication law, coordination tasks and time complexity. Sections III and IV define the rendezvous and deployment coordination tasks, respectively, and present various coordination algorithms that achieve them. For both problems, we establish the asymptotic correctness of the proposed algorithms, and characterize their time complexity. Finally, we present our conclusions in Section V. In the appendix, we review some basic computational geometric structures employed along the discussion.

Notation: We let `BooleSet` be the set $\{\texttt{true}, \texttt{false}\}$. We let $\prod_{i\in\{1,\dots,N\}} S_i$ denote the Cartesian product of sets $S_1, \dots, S_N$. We let $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote the set of strictly positive and non-negative real numbers, respectively. The set of positive natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of non-negative integers. If S is a set, then $\operatorname{diag}(S \times S) = \{(s,s) \in S \times S \mid s \in S\}$. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the floor of x. For $x \in \mathbb{R}^d$, we denote by $\|x\|_2$ and $\|x\|_\infty$ the Euclidean and the ∞-norm of x, respectively. Recall that $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\|x\|_\infty$ for all $x \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, we let $B(x,r)$ and $\overline{B}(x,r)$ denote the open and closed ball in $\mathbb{R}^d$ centered at x of radius r, respectively. We let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard orthonormal basis of $\mathbb{R}^d$. Also, we define the vectors $\mathbf{0} = (0, \dots, 0)^T$ and $\mathbf{1} = (1, \dots, 1)^T$ in $\mathbb{R}^d$. For $f, g \colon \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$ (respectively, $|f(N)| \geq k|g(N)|$ for all $N \geq N_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$. We refer the reader to Appendix I for some useful geometric concepts. Finally, we will use the notation $\operatorname{Trid}_N(a,b,c)$, $\operatorname{Circ}_N(a,b,c)$ and $\operatorname{ATrid}_N^{\pm}(a,b)$ to refer to various tridiagonal Toeplitz and circulant matrices as introduced in Appendix A of [1].

II. Synchronous robotic networks

The companion paper [1] proposes a formal model for robotic networks, and defines notions of control and communication laws, coordination tasks, and time and communication complexity. For the sake of completeness, we present here simplified versions of these notions.

Definition II.1 *A* uniform network of robotic agents *(or* robotic network*)* $\mathcal{S}$ *is a tuple* $(I, \mathcal{A}, E_{\text{cmm}})$ *consisting of*

(i) $I = \{1, \dots, N\}$*; I is called the* set of unique identifiers (UIDs)*;*

(ii) $\mathcal{A} = \{A^{[i]}\}_{i \in I}$*, with* $A^{[i]} = (X, U, X_0, f)$*, is a set of identical control systems; this set is called the* set of physical agents*;*

(iii) E_{cmm} *is a map from* $\prod_{i \in I} X$ *to the subsets of* $I \times I \setminus \operatorname{diag}(I \times I)$*; this map is called the* communication edge map. •

Definition II.2 *A* (synchronous, static, uniform, feedback) control and communication law $\mathcal{CC}$ *for* $\mathcal{S}$ *consists of the sets:*

(i) $\mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \overline{\mathbb{R}}_+$ *is an increasing sequence of time instants, called* communication schedule*;*

(ii) L *is a set containing the* `null` *element, called the* communication language*; elements of* L *are called* messages*;*

and of the maps:

(i) $\text{msg}\colon \mathbb{T} \times X \times I \to L$ *is called* message-generation function*;*

(ii) $\text{ctl}\colon \overline{\mathbb{R}}_+ \times X \times X \times L^N \to U$*,* $i \in I$*, is called* control function. •

A control and communication law $\mathcal{CC}$ is said to be *time-independent* if the message-generation and control functions are of the form $\text{msg}\colon X \times I \to L$ and $\text{ctl}\colon X \times X \times L^N \to U$, respectively.

Definition II.3 *The* evolution *of* $(\mathcal{S}, \mathcal{CC})$ *from initial conditions* $x_0^{[i]} \in {X_0}^{[i]}$*,* $i \in I$*, is the set of curves* $x^{[i],\ell}\colon [t_\ell, t_{\ell+1}] \to X$*,* $i \in I$*,* $\ell \in \mathbb{N}_0$ *satisfying*

$$\dot{x}^{[i],\ell}(t) = f\big(x^{[i],\ell}(t),\, \text{ctl}(t, x^{[i],\ell}(t), x^{[i],\ell}(t_\ell), y^{[i]}(t_\ell))\big),$$

where, for $\ell \in \mathbb{N}_0$*, and* $i \in I$*,*

$$x^{[i],\ell}(t_\ell) = x^{[i],\ell-1}(t_\ell)\,,$$

with the convention $x^{[i],-1}(t_0) = x_0^{[i]}$*. Here, the function* $y^{[i]}\colon \mathbb{T} \to L^N$ *(describing the messages received by agent* i*) has components* $y_j^{[i]}(t_\ell)$*, for* $j \in I$*, given by*

$$y_j^{[i]}(t_\ell) = \text{msg}(t_\ell, x^{[j],\ell-1}(t_\ell), i)$$

if $(i,j) \in E_{\text{cmm}}(x^{[1],\ell-1}(t_\ell), \dots, x^{[N],\ell-1}(t_\ell))$ *and* $y_j^{[i]}(t_\ell) = \texttt{null}$ *otherwise.* •

Remarks II.4 (Related concepts and notations) To distinguish between the `null` and the non-`null` messages received by an agent at a given time instant, it is convenient to define the *natural projection* $\pi_L \colon L^N \to 2^L$ that maps an array of messages y to the subset of L containing only the non-`null` messages in y.

In many uniform control and communication laws, the messages interchanged among the network agents are (quantized representations of) the agents' states. The corresponding communication language is $L = X$ and message generation function $\text{msg}_{\text{std}} \colon \mathbb{T} \times X \times I \to X$ is referred to as the *standard message-generation function* and is defined by $\text{msg}_{\text{std}}(t, x, j) = x$. •

Let us now introduce some useful examples of robotic networks. We start with a fairly common example and define some interesting variations.

Example II.5 (Locally-connected first-order agents in $\mathbb{R}^d$) Consider N points $x^{[1]}, \dots, x^{[N]}$ in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, obeying a first-order dynamics $\dot{x}^{[i]}(t) = u^{[i]}(t)$. These are identical agents of the form $A = (\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, (\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d))$. Assume that each agent can communicate to any other agent within Euclidean distance r, that is, adopt as communication edge map the r-disk proximity edge map $E_{r\text{-disk}}$ defined in Appendix I. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}} = (I, \mathcal{A}, E_{r\text{-disk}})$. •

Example II.6 (LD-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous example. For $r \in \mathbb{R}_+$, recall from Appendix I the r-limited Delaunay map $E_{r\text{-LD}}$ defined by

$$(i, j) \in E_{r\text{-LD}}(x^{[1]}, \dots, x^{[N]}) \quad \text{if and only if} \quad \big(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})\big) \cap \big(V^{[j]} \cap \overline{B}(x^{[j]}, \tfrac{r}{2})\big) \neq \emptyset, \ i \neq j,$$

where $\{V^{[1]}, \dots, V^{[N]}\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x^{[1]}, \dots, x^{[N]}\}$. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}} = (I, \mathcal{A}, E_{r\text{-LD}})$. •

Example II.7 (Locally-∞-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous two examples. For $r \in \mathbb{R}_+$, define the proximity edge map $E_{r\text{-}\infty\text{-disk}}$ by

$$(i, j) \in E_{r\text{-}\infty\text{-disk}}(x^{[1]}, \dots, x^{[N]}) \quad \text{if and only if} \quad \|x^{[i]} - x^{[j]}\|_\infty \leq r, \ i \neq j.$$

These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}} = (I, \mathcal{A}, E_{r\text{-}\infty\text{-disk}})$. •

In order to analyze the performance of a communication and control law, we first define the notion of coordination task, and of task achievement by a robotic network.

Definition II.8 (Coordination task) *Let $\mathcal{S}$ be a robotic network. A* (static) coordination task *for $\mathcal{S}$ is a map $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \to$ `BooleSet`. Additionally, let $\mathcal{CC}$ a control and communication law for $\mathcal{S}$. The law $\mathcal{CC}$* achieves *the task $\mathcal{T}$ if for all initial conditions $x_0^{[i]} \in X_0^{[i]}$, $i \in I$, the corresponding network evolution $t \mapsto x(t)$ has the property that there exists $T \in \mathbb{R}_+$ such that $\mathcal{T}(x(t)) =$ `true` for all $t \geq T$.* •

The notions of time and communication complexity describe the performance and cost of a control and communication law completing a certain coordination task. Here, we focus on time complexity.

Definition II.9 (Time complexity) *Let $\mathcal{S}$ be a robotic network, let $\mathcal{T}$ be a coordination task for $\mathcal{S}$ and let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$.*

(i) *The* time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from $x_0 \in \prod_{i\in I} X_0^{[i]}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) = \inf\{\ell \mid \mathcal{T}(x(t_k)) = \texttt{true}\,,\ \textit{for all } k \geq \ell\}\,,$$

where $t \mapsto (x(t))$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from x_0.

(ii) *The* time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}) = \sup\left\{\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) \mid x_0 \in \prod_{i\in I} X_0^{[i]}\right\}.$$

(iii) *The* time complexity of $\mathcal{T}$ *is*

$$\mathrm{TC}(\mathcal{T}) = \inf\{\mathrm{TC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \textit{ achieves } \mathcal{T}\}.$$ •

III. RENDEZVOUS

In this section, we introduce rendezvous coordination tasks and analyze various coordination algorithms that achieve them, providing upper and lower bounds on their time complexity. Along the section, we will consider the networks $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ presented in Example II.5 and Example II.6, respectively.

A. Rendezvous tasks

First, let $\mathcal{S} = (I, \mathcal{A}, E_{\text{cmm}})$ be a uniform robotic network. The *(exact) rendezvous task* $\mathcal{T}_{\text{rndzvs}}\colon X^N \to$ `BooleSet` for $\mathcal{S}$ is the static task defined by

$$\mathcal{T}_{\text{rndzvs}}(x^{[1]}, \dots, x^{[N]}) = \begin{cases} \texttt{true}, & \text{if } x^{[i]} = x^{[j]}, \text{ for all } (i,j) \in E_{\text{cmm}}(x^{[1]}, \dots, x^{[N]}), \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

$\mathcal{T}_{\text{rndzvs}}(x^{[1]},\dots,x^{[N]}) = \texttt{true}$ if and only if

$$x^{[i]} = x^{[j]}, \text{ for all } (i,j) \in E_{\text{cmm}}(x^{[1]},\dots,x^{[N]}).$$

Second, let $\mathcal{S} = (I, \mathcal{A}, E_{\text{cmm}})$ be a uniform robotic network with agents' state space $X \subset \mathbb{R}^d$. Examples networks of this form are $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$, see Examples II.5 and III-B, and $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$, see Examples II.6. For $\varepsilon > 0$, the *ε-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}} \colon X^N \to \texttt{BooleSet}$ for $\mathcal{S}$ is defined by $\mathcal{T}_{\varepsilon\text{-rndzvs}}(x) = \texttt{true}$ if and only if

$$\Big\| x^{[i]} - \operatorname{avrg}\big(\{x^{[i]}\} \cup \{x^{[j]} \mid (i,j) \in E_{\text{cmm}}(x)\}\big) \Big\|_2 < \varepsilon,$$

for all $i \in I$. Here $x = (x^{[1]},\dots,x^{[N]}) \in X^N \subset (\mathbb{R}^d)^N$. In other words, $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is $\texttt{true}$ at $x \in (\mathbb{R}^d)^N$ if, for all $i \in I$, $x^{[i]}$ is at distance less than ε from the average of its own position with the position of its E_{cmm}-neighbors.

B. Rendezvous without connectivity via the move-toward-average control and communication law

From Example II.5, consider the uniform network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{R}^d$. We now define a static, uniform and time-independent law that we refer to as the move-toward-average law and that we denote by $\mathcal{CC}_{\text{avrg}}$. We loosely describe it as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent transmits its position. Between communication rounds, each agent moves towards and reaches the point that is the average of its neighbors' positions; the average point is computed including the agent's own position.

Next, we *formally* define the law as follows. First, we take $\mathbb{T} = \mathbb{N}_0$ and we assume that each agent operates with the standard message-generation function, i.e., we set $L = \mathbb{R}^d$ and $\operatorname{msg}(x,j) = \operatorname{msg}_{\text{std}}(x,j) = x$. Second, we define the control function $\operatorname{ctl} \colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\operatorname{ctl}(x, x_{\text{smpld}}, y) = -k_{\text{prop}} \operatorname{vers}\big(x - \operatorname{avrg}(y \cup \{x_{\text{smpld}}\})\big),$$

where $k_{\text{prop}} \geq r$, $\operatorname{vers} \colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\operatorname{vers}(0) = 0$ and $\operatorname{vers}(v) = v/\|v\|_2$ for $v \neq 0$, and the map avrg computes the average of a finite point set in $\mathbb{R}^d$:

$$\operatorname{avrg}(S) = \frac{1}{\displaystyle\sum_{p \in \pi_{\mathbb{R}}(S)} 1} \sum_{p \in \pi_{\mathbb{R}}(S)} p\,.$$

In summary we set $\mathcal{CC}_{\text{avrg}} = (\mathbb{N}_0, \mathbb{R}^d, \text{msg}_{\text{std}}, \text{ctl})$. An implementation of this control and communication law is shown in Fig. 1 for $d = 1$. Note that, along the evolution, (1) several agents *rendezvous*, i.e., agree upon a common location, and (2) some agents are connected at the simulation's beginning and not connected at the simulation's end. Finally, we remark that this law is related to the Vicsek's model discussed in [15], [22]. •

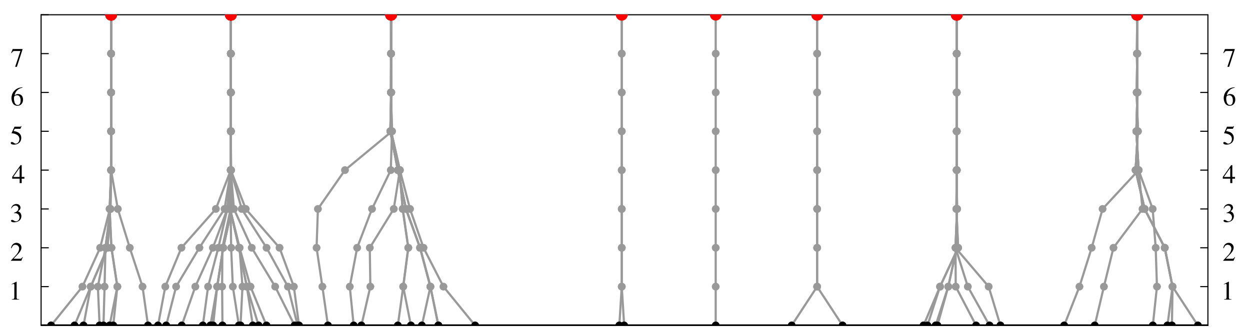

Fig. 1. Evolution of a robotic network under the move-toward-average control and communication law in Example III-B implemented over the r-disk graph, with $r = 1.5$. The vertical axis corresponds to the elapsed time, and the horizontal axis to the positions of the agents in the real line. The 51 agents are initially randomly deployed over the interval $[-15, 15]$.

The next result characterizes the complexity of this law.

Theorem III.1 *For* $d = 1$*, the network* $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$*, the law* $\mathcal{CC}_{\text{avrg}}$*, and the task* $\mathcal{T}_{\text{rndzvs}}$ *satisfy* $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in O(N^5)$ *and* $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in \Omega(N)$.

Proof: One can easily prove that, along the evolution of the network, the ordering of the agents is preserved, i.e., if $x^{[i]}(\ell) \leq x^{[j]}(\ell)$, then $x^{[i]}(\ell+1) \leq x^{[j]}(\ell+1)$. However, links between agents are not necessarily preserved (see e.g. Figure 1). Indeed, connected components may split along the evolution. However, mergings are not possible. Consider two contiguous connected components C_1 and C_2, with C_1 to the left of C_2. By definition, the rightmost agent of C_1 and the leftmost agent of C_2 are at a distance strictly bigger than r. Now, by executing the algorithm, they can only but increase that distance, since the rightmost agent of C_1 will move to the left, and the leftmost agent of C_2 will move to the right. Therefore, connected components do not merge.

Consider first the case of an initial configuration of the network for which the communication graph remains connected throughout the evolution. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. Let $\alpha \in \{3, \ldots, N\}$ have the property that agents $\{2,\dots,\alpha-1\}$ are neighbors of agent 1, and agent α is not. (If instead all agents are within an interval of length r, then rendezvous is achieved in 1 time instant, and the statement in theorem is easily seen to be true.) Note that we can assume that agents $\{2,\dots,\alpha-1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent α, rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$x^{[1]}(1) = \frac{1}{\alpha-1}\sum_{k=1}^{\alpha-1} x^{[k]}(0), \qquad x^{[\gamma]}(1) \in \Big[\frac{1}{\alpha}\sum_{k=1}^{\alpha} x^{[k]}(0), *\Big],\ \gamma \in \{2,\dots,\alpha-1\},$$

where $*$ denotes certain unimportant point.

Now, we show that

$$x^{[1]}(\alpha-1) - x^{[1]}(0) \geq \frac{r}{\alpha(\alpha-1)}. \tag{1}$$

Let us first show the inequality for $\alpha = 3$. Note that the fact that the communication graph remains connected implies that agent 2 is still a neighbor of agent 1 at the time instant $\ell = 1$. Therefore $x^{[1]}(2) \geq \frac{1}{2}(x^{[1]}(1)+x^{[2]}(1))$, and from here we deduce

$$\begin{aligned} x^{[1]}(2) - x^{[1]}(0) &\geq \frac{1}{2}\big(x^{[2]}(1) - x^{[1]}(0)\big) \\ &\geq \frac{1}{2}\Big(\frac{1}{3}\big(x^{[1]}(0)+x^{[2]}(0)+x^{[3]}(0)\big) - x^{[1]}(0)\Big) \geq \frac{1}{6}\big(x^{[3]}(0) - x^{[1]}(0)\big) \geq \frac{r}{6}. \end{aligned}$$

Let us now proceed by induction. Assume that inequality (1) is valid for $\alpha-1$, and let us prove it for α. Consider first the possibility when at the time instant $\ell = 1$, the agent $\alpha-1$ is still a neighbor of agent 1. In this case, $x^{[1]}(2) \geq \frac{1}{\alpha-1}\sum_{k=1}^{\alpha-1} x^{[k]}(1)$, and from here we deduce

$$\begin{aligned} x^{[1]}(2) - x^{[1]}(0) &\geq \frac{1}{\alpha-1}\Big(x^{[\alpha-1]}(1) - x^{[1]}(0)\Big) \geq \frac{1}{\alpha-1}\Big(\frac{1}{\alpha}\sum_{k=1}^{\alpha} x^{[k]}(0) - x^{[1]}(0)\Big) \\ &\geq \frac{1}{\alpha(\alpha-1)}\Big(x^{[\alpha]}(0) - x^{[1]}(0)\Big) \geq \frac{r}{\alpha(\alpha-1)}, \end{aligned}$$

which in particular implies (1). Consider then the case when agent $\alpha-1$ is not a neighbor of agent 1 at the time instant $\ell = 1$. Let $\beta < \alpha$ such that agent $\beta-1$ is a neighbor of agent 1 at $\ell = 1$, but agent β is not. Since $\beta < \alpha$, we have by induction $x^{[1]}(\beta) - x^{[1]}(1) \geq \frac{r}{\beta(\beta-1)}$. From here, we deduce that $x^{[1]}(\alpha-1) - x^{[1]}(0) \geq \frac{r}{\alpha(\alpha-1)}$.

Inequality (1) implies that, at most in $\alpha-1 \leq N-1$ time instants, the leftmost agent traverses a distance greater than or equal to $\frac{r}{N(N-1)}$ (provided that at each step there exists at least another agent which is not its neighbor).

Since $\operatorname{diam}(x_0, I) \leq (N-1)r$, we deduce that in $N(N-1)^3$ time instants there cannot be any agent which is not a neighbor of the agent 1. Hence, all agents rendezvous at the next time instant. Consequently,

$$\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}, x_0) \leq N(N-1)^3 + 1.$$

Finally, for a general initial configuration x_0, because there are a finite number of agents, only a finite number of splittings (at most $N-1$) of the connected components of the communication graph can take place along the evolution. Therefore, we conclude $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) = O(N^5)$.

Let us now prove the lower bound. Consider an initial configuration $x_0 \in \mathbb{R}^N$ where all agents are positioned in increasing order according to their identity, and exactly at a distance r apart, say $(x_0)_{i+1} - (x_0)_i = r$, $i \in \{1, \ldots, N-1\}$. Assume for simplicity that N is odd - when N is even, one can reason in an analogous way. Because of the symmetry of the initial condition, in the first time step, only agents 1 and N move. All the remaining agents remain in their position because it coincides with the average of its neighbors' position and its own. At the second time step, only agents 1, 2, $N-1$ and N move, and the others remain still because of the symmetry. Applying this idea iteratively, one deduces the time step when agents $\frac{N-1}{2}$ and $\frac{N+3}{2}$ move for the first time is lower bounded by $\frac{N-1}{2}$. Since both agents have still at least a neighbor (agent $\frac{N+1}{2}$), the task $\mathcal{T}_{\text{rndzvs}}$ has not been achieved yet at this time step. Therefore, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}, x_0) \geq \frac{N-1}{2}$, and the result follows. ■

C. Rendezvous with connectivity constraint via the circumcenter control and communication law

Here we define the *circumcenter* control and communication law $\mathcal{CC}_{\text{crcmcntr}}$ for both networks $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$. This is a uniform, static, time-independent law originally introduced by [4] and later studied in [6], [7]. Loosely speaking, the evolution of the network under the circumcenter control and communication law can be described as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself, and (iii) it moves toward this circumcenter while maintaining connectivity with its neighbors.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\mathrm{msg}^{[i]} = \mathrm{msg}_{\text{std}}$, $i \in I$. In order to define the control function, we need to introduce some preliminary constructions. First, connectivity is maintained by restricting the allowable motion of each agent in the following appropriate manner. If agents i and j are neighbors at time $\ell \in \mathbb{N}_0$, then we require their subsequent positions to belong to $\overline{B}(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2})$. If an agent i has its neighbors at locations $\{q_1, \dots, q_l\}$ at time ℓ, then its *constraint set* $\mathcal{D}_{x^{[i]}(\ell),r}(\{q_1, \dots, q_l\})$ is

$$\mathcal{D}_{x^{[i]}(\ell),r}(\{q_1, \dots, q_l\}) = \bigcap_{q \in \{q_1,\dots,q_l\}} \overline{B}\Big(\frac{x^{[i]}(\ell)+q}{2}, \frac{r}{2}\Big).$$

Second, in order to maximize the displacement toward the circumcenter of the point set comprised of its neighbors and of itself, each agent solves a convex optimization problem that can be stated in general as follows. For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda(q_0, q_1, Q)$ denote the solution to the strictly convex problem:

$$\text{maximize } \lambda$$

$$\text{subject to } \lambda \leq 1, \ (1-\lambda)q_0 + \lambda q_1 \in Q.$$

Under the stated assumptions the solution exists and is unique. Third, note that since the agents operate with the standard message-generation function, it is clear that the natural projection $\pi_{\mathbb{R}^d}$ maps the messages $y^{[i]}(\ell)$ received at time $\ell \in \mathbb{N}_0$ by the agent $i \in I$ onto the positions of its neighbors. We are now ready to define the last constitutive element of $\mathcal{CC}_{\text{crcmcntr}}$. Define the control function ctl: $\mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \lambda_* \cdot (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}) - x_{\text{smpld}}), \tag{2}$$

with $\lambda_* = \lambda(x_{\text{smpld}}, (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}), \mathcal{D}_{x_{\text{smpld}},r}(\pi_{\mathbb{R}^d}(y)))$. Evolving under this control law, it is clear that, at time $\lfloor t \rfloor + 1$, each agent i reaches the point $(1-\lambda_*)x^{[i]}(\lfloor t \rfloor) + \lambda_* \, \text{Circum}(\pi_{\mathbb{R}^d}(y^{[i]}(\lfloor t \rfloor)) \cup \{x^{[i]}(\lfloor t \rfloor)\})$.

Next, we consider the network $\mathcal{S}_{r\text{-}\infty\text{-disk}}$ of locally-∞-connected first-order agents in $\mathbb{R}^d$, see Example II.7. For this network we define the *parallel circumcenter law*, $\mathcal{CC}_{\text{pll-crcmcntr}}$, by designing d decoupled circumcenter laws running in parallel on each coordinate axis of $\mathbb{R}^d$. As before, this law is uniform, static and time-independent. As before, we set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. We define the control function ctl: $\mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \Big(\text{Circum}(\tau_1(\mathcal{M})) - \tau_1(x_{\text{smpld}}), \dots, \text{Circum}(\tau_d(\mathcal{M})) - \tau_d(x_{\text{smpld}}) \Big), \tag{3}$$

where $\mathcal{M} = \pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}$, and $\tau_1, \dots, \tau_d \colon \mathbb{R}^d \to \mathbb{R}$ denote the canonical projections of $\mathbb{R}^d$ onto $\mathbb{R}$. See Fig. 2 for an illustration of this law in $\mathbb{R}^2$.

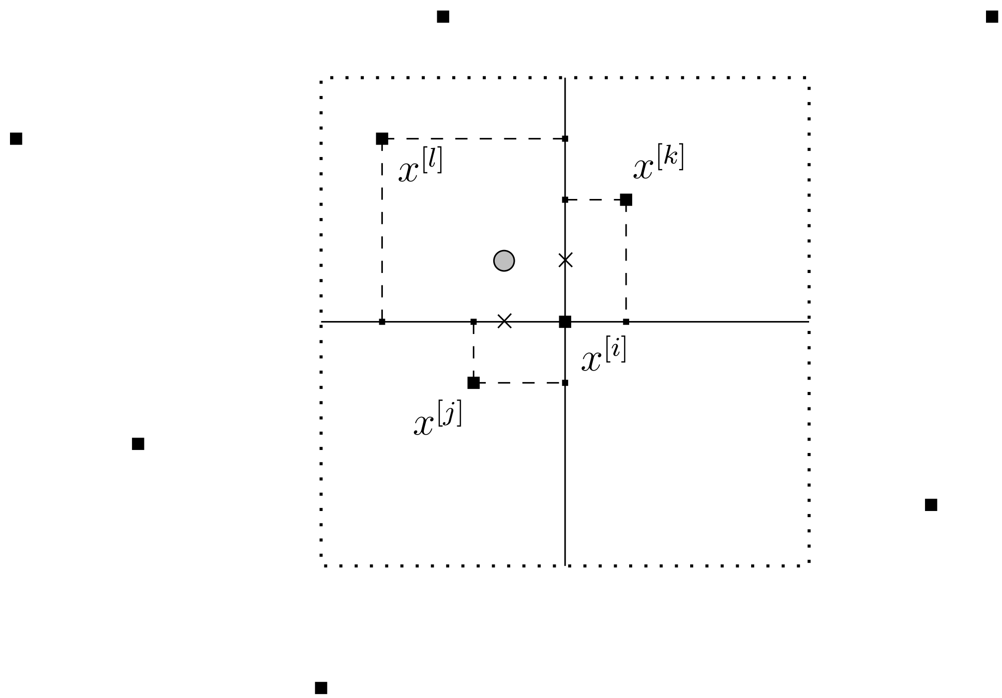

Fig. 2. Parallel circumcenter control and communication law in $\mathbb{R}^2$. The target point for the agent i is plotted in light gray and has coordinates $(\mathrm{Circum}(\tau_1(\mathcal{M}^{[i]})), \mathrm{Circum}(\tau_2(\mathcal{M}^{[i]})))$.

Asymptotic behavior and complexity analysis: The following theorem summarizes the results known in the literature about the asymptotic properties of the circumcenter law.

Theorem III.2 (Correctness of the circumcenter law) *For $d \in \mathbb{N}$, $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, the following statements hold:*

(i) *on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;*

(ii) *on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;*

(iii) *on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}}$, the law $\mathcal{CC}_{\text{pll-crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;*

(iv) *the evolutions of $(\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}, \mathcal{CC}_{\text{crcmcntr}})$, of $(\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}, \mathcal{CC}_{\text{crcmcntr}})$, and of $(\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}}, \mathcal{CC}_{\text{pll-crcmcntr}})$ have the property that, if two agents belong to the same connected component of the communication graph at $\ell \in \mathbb{N}_0$, then they continue to belong to the same connected component of the communication graph for all subsequent times $k \geq \ell$.* •

Proof: The results on $\mathcal{S}_{r\text{-disk}}$ appeared originally in [4]. The proof for the results on $\mathcal{S}_{r\text{-LD}}$ is provided in [7]. We postpone the proof for $\mathcal{S}_{r\text{-}\infty\text{-disk}}$ to the proof of Theorem III.3 below. ■

Next we analyze the time complexity of $\mathcal{CC}_{\text{crcmcntr}}$. We provide complete results for the case $d = 1$. As we see next, the complexity of $\mathcal{CC}_{\text{crcmcntr}}$ differs dramatically when applied to the two robotic networks with different communication graphs.

Theorem III.3 (Time complexity of circumcenter law) *For $r \in \mathbb{R}_+$ and $\varepsilon \in]0, 1[$, the following statements hold:*

(i) *for $d = 1$, on the network $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N)$;*

(ii) *for $d = 1$, on the network $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N^2 \log(N\varepsilon^{-1}))$;*

(iii) *for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{pll-crcmcntr}}) \in \Theta(N)$.* •

Proof: Let $x_0 \in \mathbb{R}^N$. Throughout the proof, neighboring relationships are understood with respect to the r-disk graph. First of all, let us show that, for $n = 1$, the connectivity constraints on each agent $i \in I$ imposed by the constraint set $\mathcal{D}_{x^{[i]},r}(\pi_{\mathbb{R}}(y)))$ are superfluous, i.e., the solution of the convex optimization problem is $\lambda_* = 1$ (cf. equation (2)). To see this, assume that agents i and j are neighbors at time instant ℓ, define $\mathcal{M}^{[i]}$ as $\pi_{\mathbb{R}^d}(y^{[i]}(\ell)) \cup \{x^{[i]}(\ell)\}$, and let us show that $\mathrm{Circum}(\mathcal{M}^{[i]})$ belongs to $\overline{B}(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2})$. Without loss of generality, let $x^{[i]}(\ell) \leq x^{[j]}(\ell)$. Let $x_-^{[i]}(\ell)$, $x_+^{[i]}(\ell)$ denote the positions of the leftmost and rightmost agents among the neighbors of agent i. Note that $x^{[i]}(\ell) \leq x^{[j]}(\ell) \leq x_+^{[i]}(\ell)$ and $\mathrm{Circum}(\mathcal{M}^{[i]}) = \frac{1}{2}(x_-^{[i]}(\ell) + x_+^{[i]}(\ell))$. Then,

$$\begin{aligned}\left|\,\mathrm{Circum}(\mathcal{M}^{[i]}) - \frac{1}{2}(x^{[i]}(\ell) + x^{[j]}(\ell))\right| &= \frac{1}{2}\left|x_-^{[i]}(\ell) - x^{[i]}(\ell) + x_+^{[i]}(\ell) - x^{[j]}(\ell)\right| \\ &\leq \frac{1}{2}\max\{|x_-^{[i]}(\ell) - x^{[i]}(\ell)|, |x_+^{[i]}(\ell) - x^{[j]}(\ell)|\} \leq \frac{r}{2}\,,\end{aligned}$$

as claimed. Therefore, we have that $x^{[i]}(\ell + 1) = \mathrm{Circum}(\mathcal{M}^{[i]})$. Likewise, one can deduce $\mathrm{Circum}(\mathcal{M}^{[i]}) \leq \mathrm{Circum}(\mathcal{M}^{[j]})$, and therefore, the order of the agents is preserved.

Consider first the case when $E_{r\text{-disk}}(x_0)$ is connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. Let $\alpha \in \{3, \ldots, N\}$ have the property that agents $\{2, \ldots, \alpha - 1\}$ are neighbors of agent 1, and agent α is not. (If instead all agents are within an interval of length r, then rendezvous is achieved in 1 time instant, and the statement in theorem is easily seen to be true.) See Fig. 3 for an illustration of these definitions. Note that we can assume that agents $\{2, \ldots, \alpha - 1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors

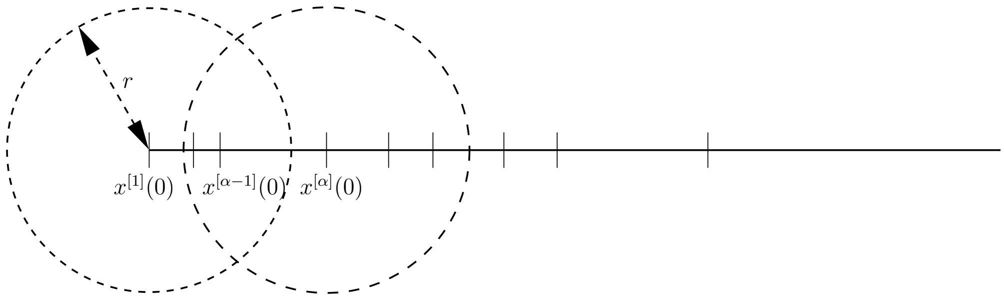

Fig. 3. Definition of $\alpha \in \{3, \dots, N\}$ for an initial network configuration.

of agent 1 and not of agent α, rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$x^{[1]}(1) = \frac{x^{[1]}(0) + x^{[\alpha-1]}(0)}{2}, \quad x^{[\gamma]}(1) \in \left[\frac{x^{[1]}(0) + x^{[\alpha]}(0)}{2}, \frac{x^{[1]}(0) + x^{[\gamma]}(0) + r}{2}\right], \ \gamma \in \{2, \dots, \alpha - 1\}.$$

These equalities imply that $x^{[1]}(1) - x^{[1]}(0) = \frac{1}{2}\big(x^{[\alpha-1]}(0) - x^{[1]}(0)\big) \leq \frac{1}{2}r$. Analogously, we deduce $x^{[1]}(2) - x^{[1]}(1) \leq \frac{1}{2}r$, and therefore

$$x^{[1]}(2) - x^{[1]}(0) \leq r. \tag{4}$$

On the other hand, from $x^{[1]}(2) \in \big[\frac{1}{2}\big(x^{[1]}(1) + x^{[\alpha-1]}(1)\big), *\big]$ (where the symbol $*$ represents a certain unimportant point in $\mathbb{R}$), we deduce that

$$\begin{aligned} x^{[1]}(2) - x^{[1]}(0) &\geq \frac{1}{2}\big(x^{[1]}(1) + x^{[\alpha-1]}(1)\big) - x^{[1]}(0) \geq \frac{1}{2}\big(x^{[\alpha-1]}(1) - x^{[1]}(0)\big) \\ &\geq \frac{1}{2}\Big(\frac{x^{[1]}(0) + x^{[\alpha]}(0)}{2} - x^{[1]}(0)\Big) = \frac{1}{4}\big(x^{[\alpha]}(0) - x^{[1]}(0)\big) \geq \frac{1}{4}r. \end{aligned} \tag{5}$$

Inequalities (4) and (5) mean that, after at most two time instants, agent 1 has traveled an amount larger than $r/4$. In turn this implies that

$$\frac{\operatorname{diam}(x_0, I)}{r} \leq \operatorname{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}, x_0) \leq \frac{4 \operatorname{diam}(x_0, I)}{r}.$$

If $E_{r\text{-disk}}(x_0)$ is not connected, note that along the network evolution, the connected components of the r-disk graph do not change. Therefore, using the previous characterization on the amount traveled by the leftmost agent of each connected component in at most two time instants, we deduce that

$$\frac{1}{r} \max_{C \in \mathcal{C}_{E_{r\text{-disk}}(x_0)}} \operatorname{diam}(x_0, C) \leq \operatorname{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}, x_0) \leq \frac{4}{r} \max_{C \in \mathcal{C}_{E_{r\text{-disk}}(x_0)}} \operatorname{diam}(x_0, C).$$

Note that the connectedness of each $C \in \mathcal{C}_{E_{r\text{-disk}}}(x_0)$ implies that $\operatorname{diam}(x_0, C) \le (N-1)r$, and therefore $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in O(N)$. Moreover, for $x_0 \in \mathbb{R}^N$ such that $(x_0)_{i+1} - (x_0)_i = r$, $i \in \{1, \dots, N-1\}$, we have $\operatorname{diam}(x_0, I) = (N-1)r$, and therefore $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}, x_0) \ge N-1$. This concludes the proof of fact (i):

$$\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N)\,.$$

Next we prove fact (ii). In the r-limited Delaunay graph, two agents on the line that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Also, note that the r-limited Delaunay graph and the r-disk graph have the same connected components (cf. [12]). Using an argument similar to the one above, one can show that the connectivity constraints imposed by the constraint sets set $\mathcal{D}_{x^{[i]}(\lfloor t \rfloor), r}(\pi_{\mathbb{R}}(y)))$ are again superfluous.

Consider first the case when $E_{r\text{-LD}}(x_0)$ is connected. Note that this is equivalent to $E_{r\text{-disk}}(x_0)$ being connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \le \dots \le x^{[N]}(0) = (x_0)_N$. The evolution of the network under $\mathcal{CC}_{\text{crcmcntr}}$ can then be described as the discrete-time dynamical system

$$x^{[1]}(\ell+1) = \frac{1}{2}(x^{[1]}(\ell) + x^{[2]}(\ell))\,, \quad x^{[2]}(\ell+1) = \frac{1}{2}(x^{[1]}(\ell) + x^{[3]}(\ell))\,, \ \dots\,,$$
$$\,,\ \dots\,,\ x^{[N-1]}(\ell+1) = \frac{1}{2}(x^{[N-2]}(\ell) + x^{[N]}(\ell))\,, \quad x^{[N]}(\ell+1) = \frac{1}{2}(x^{[N-1]}(\ell) + x^{[N]}(\ell))\,.$$

Note that this evolution respects the ordering of the agents. Equivalently, we can write $x(\ell+1) = A\,x(\ell)$, where A is the $N \times N$ matrix given by

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \dots & \dots & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Note that $A = \mathrm{ATrid}_N^+\left(\frac{1}{2}, 0\right)$ as defined in [1, Appendix A]. Theorem A.4(i) in [1] implies that, for $x_{\text{ave}} = \frac{1}{N}\mathbf{1}^T x(0)$, we have that $\lim_{\ell \to +\infty} x(\ell) = x_{\text{ave}}\mathbf{1}$, and that the maximum time required for $\left\|x(\ell) - x_{\text{ave}}\mathbf{1}\right\|_2 \le$

$\eta\|x(0) - x_{\text{ave}}\mathbf{1}\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \eta^{-1}\big)$. (As an aside, this also implies that the agents rendezvous at the location given by the average of their initial positions. In other words, we can forecast the asymptotic rendezvous position for this case, as opposed to the case with the r-disk communication graph.)

Next, let us convert the contraction inequality on 2-norms into an appropriate inequality on ∞-norms. Note that $\operatorname{diam}(x_0, I) \leq (N-1)r$ because $E_{r\text{-LD}}(x_0)$ is connected. Therefore

$$\|x(0) - x_{\text{ave}}\mathbf{1}\|_\infty = \max_{i \in I} |x^{[i]}(0) - x_{\text{ave}}| \leq |x_0^{[1]} - x_0^{[N]}| \leq (N-1)r.$$

For ℓ of order $N^2 \log \eta^{-1}$, we use this bound on $\|x(0) - x_{\text{ave}}\mathbf{1}\|_\infty$ and the basic inequalities $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{N}\|v\|_\infty$ for all $v \in \mathbb{R}^N$, to obtain:

$$\|x(\ell) - x_{\text{ave}}\mathbf{1}\|_\infty \leq \|x(\ell) - x_{\text{ave}}\mathbf{1}\|_2 \leq \eta\|x(0) - x_{\text{ave}}\mathbf{1}\|_2 \leq \eta\sqrt{N}\|x(0) - x_{\text{ave}}\mathbf{1}\|_\infty \leq \eta\sqrt{N}(N-1)r.$$

This means that $(r\varepsilon)$-rendezvous is achieved for $\eta\sqrt{N}(N-1)r = r\varepsilon$, that is, in time $O(N^2 \log \eta^{-1}) = O(N^2 \log(N\varepsilon^{-1}))$. Next, we show the lower bound. Consider the unit-length eigenvector $\mathbf{v}_N = \sqrt{\frac{2}{N+1}}(\sin\frac{\pi}{N+1}, \dots, \sin\frac{N\pi}{N+1})^T \in \mathbb{R}^N$ of $\operatorname{Trid}_{N-1}(\frac{1}{2}, 0, \frac{1}{2})$ corresponding to the largest singular value $\cos(\frac{\pi}{N})$. This vector is an eigenvector of $\operatorname{Trid}_{N-1}(\frac{1}{2}, 0, \frac{1}{2})$ corresponding to the largest singular value $\cos(\frac{\pi}{N})$. For $\mu = \frac{-1}{10\sqrt{2}} r N^{5/2}$, we then define the initial condition $x_0 = \mu P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{N-1} \end{bmatrix} \in \mathbb{R}^N$. One can show that $(x_0)_i < (x_0)_{i+1}$ for $i \in \{1, \dots, N-1\}$, that $(x_0)_{\text{ave}} = 0$, and that $\max\{(x_0)_{i+1} - (x_0)_i \mid i \in \{1, \dots, N-1\}\} \leq r$. Using [1, Lemma A.5] and because $\|w\|_\infty \leq \|w\|_2 \leq \sqrt{N}\|w\|_\infty$ for all $w \in \mathbb{R}^N$, we compute

$$\|x_0\|_\infty = \frac{rN^{5/2}}{10\sqrt{2}} \Big\| P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{N-1} \end{bmatrix} \Big\|_\infty \geq \frac{rN^2}{10\sqrt{2}} \Big\| P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{N-1} \end{bmatrix} \Big\|_2 \geq \frac{rN}{10\sqrt{2}} \|\mathbf{v}_{N-1}\|_2 = \frac{rN}{10\sqrt{2}}.$$

The trajectory $x(\ell) = (\cos(\frac{\pi}{N}))^\ell x_0$ therefore satisfies

$$\|x(\ell)\|_\infty = \Big(\cos\Big(\frac{\pi}{N}\Big)\Big)^\ell \|x_0\|_\infty \geq \frac{rN}{10\sqrt{2}} \Big(\cos\Big(\frac{\pi}{N}\Big)\Big)^\ell.$$

Therefore, $\|x(\ell)\|_\infty$ is larger than $\frac{1}{2}r\varepsilon$ so long as $\frac{1}{10\sqrt{2}}N(\cos(\frac{\pi}{N}))^\ell > \frac{1}{2}\varepsilon$, that is, so long as

$$\ell < \frac{\log(\varepsilon^{-1}N) - \log(5\sqrt{2})}{-\log\big(\cos(\frac{\pi}{N})\big)}.$$

The rest of the proof is analogous to the one of Theorem A.3(i) in [1] for the lower bound result.

If $E_{r\text{-LD}}(x_0)$ is not connected, note that along the network evolution, the connected components do not change. Therefore, the previous reasoning can be applied to each connected component. Since the number of agents in each connected component is strictly less that N, the time complexity can only but improve. Therefore, we conclude that

$$\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N^2 \log(N\varepsilon^{-1})) \,.$$

This completes the proof of fact (ii).

Finally, we prove the statements regarding $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$ and $\mathcal{CC}_{\text{pll-crcmcntr}}$ in fact (iii) and in the previous Theorem III.2. By definition, agents i and j are neighbors at time $\ell \in \mathbb{N}_0$ if and only if $\|x^{[i]}(\ell) - x^{[j]}(\ell)\|_\infty \leq r$, which is equivalent to

$$|\tau_k(x^{[i]}(\ell)) - \tau_k(x^{[j]}(\ell))| \leq r \,, \quad k \in \{1, \dots, d\} \,.$$

Recall from the proof of fact (i) that the connectivity constraints of $\mathcal{CC}_{\text{crcmcntr}}$ on each agent are trivially satisfied in the 1-dimensional case. This fact has the following important consequence: from the expression for the control function in $\mathcal{CC}_{\text{pll-crcmcntr}}$, we deduce that the evolution under $\mathcal{CC}_{\text{pll-crcmcntr}}$ of the robotic network $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$ (in d dimensions) can be alternatively described as the evolution under $\mathcal{CC}_{\text{crcmcntr}}$ of d robotic networks $\mathcal{S}_{\mathbb{R},r\text{-disk}}$ in $\mathbb{R}$. The correctness and the time complexity results now follows from the analysis of $\mathcal{CC}_{\text{crcmcntr}}$ at $d = 1$. ∎

Remark III.4 Theorem III.3 induces a lower bound on the time communication complexity of the circumcenter law for the higher-dimensional case. Indeed, as a consequence of this result, we have

(i) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Omega(N)$;

(ii) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Omega(N^2 \log(N\varepsilon^{-1}))$.

We have performed extensive numerical simulations for the case $d = 2$ and the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$. We have ran the algorithm starting from generic initial configurations (where, in particular, agents' positions are not aligned) contained in a bounded region of $\mathbb{R}^2$. We have consistently obtained that the time complexity to achieve $\mathcal{T}_{\text{rndzvs}}$ with $\mathcal{CC}_{\text{crcmcntr}}$ starting from these initial configurations is independent of the number of agents. This leads us to conjecture that, in fact, initial configurations where all agents are aligned (i.e., the 1-dimensional case) give rise to the worst possible performance of the algorithm. In more formal terms, we conjecture that, for $d \geq 2$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) = \Theta(N)$. •

IV. DEPLOYMENT

In this section, we introduce the deployment coordination task and analyze a coordination algorithm that achieves it, providing upper and lower bounds on its time complexity. Along the section, we consider the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ presented in Example II.6 with parameter $r \in \mathbb{R}_+$. We assume we are given a convex simple polytope $Q \subset \mathbb{R}^d$, with an integrable density function $\phi\colon Q \to \mathbb{R}_+$. We assume that the initial positions of the agents belong to Q and we intend to design a control law that keeps them in Q for subsequent times.

A. Deployment task

By optimal deployment on the convex simple polytope $Q \subset \mathbb{R}^d$ with density function $\phi\colon Q \to \mathbb{R}_+$, we mean the following objective: place the agents on Q so that the expected square Euclidean distance from any point in Q to one of the agents is minimized. To define this task formally, let us review some known preliminary notions; we will require some computational geometric notions from Appendix I. We consider the following network objective function $\mathcal{H}_{\text{deplmnt}}\colon Q^N \to \mathbb{R}$,

$$\mathcal{H}_{\text{deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \|q - x^{[i]}\|_2^2 \, \phi(q) dq \,. \tag{6}$$

This function and variations of it are studied in the facility location and resource allocation research literature; see [23], [11]. It is convenient [12] to study a generalization of this function. For $r \in \mathbb{R}_+$, define the saturation function $\operatorname{sat}_r\colon \mathbb{R} \to \mathbb{R}$ by $\operatorname{sat}_r(x) = x$ if $x \leq r$ and $\operatorname{sat}_r(x) = r$ otherwise. For $r \in \mathbb{R}_+$, define the new objective function $\mathcal{H}_{r\text{-deplmnt}}\colon Q^N \to \mathbb{R}$ by

$$\mathcal{H}_{r\text{-deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \operatorname{sat}_{\frac{r}{2}}(\|q - x^{[i]}\|_2^2) \, \phi(q) dq \,. \tag{7}$$

Note that if $r \geq 2 \operatorname{diam}(Q)$, then $\mathcal{H}_{\text{deplmnt}} = \mathcal{H}_{r\text{-deplmnt}}$. Let $\{V^{[1]}, \dots, V^{[N]}\}$ be the Voronoi partition of Q associated with $\{x^{[1]}, \dots, x^{[N]}\}$. The partial derivative of the cost function takes the following meaningful form

$$\frac{\partial \mathcal{H}_{r\text{-deplmnt}}}{\partial x^{[i]}}(x^{[1]}, \dots, x^{[N]}) = 2 \operatorname{Mass}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) \cdot \big(\operatorname{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) - x^{[i]}\big) \,, \quad i \in I \,.$$

(Here, as in Appendix I, $\operatorname{Mass}(S)$ and $\operatorname{Centroid}(S)$ are, respectively, the mass and the centroid of $S \subset \mathbb{R}^d$.) Clearly, the critical points of $\mathcal{H}_{r\text{-deplmnt}}$ are network states where $x^{[i]} = \operatorname{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2}))$. We call such configurations $\frac{r}{2}$-centroidal Voronoi configurations. For $r \geq 2 \operatorname{diam}(Q)$, they coincide with the standard centroidal Voronoi configurations on Q. Fig. 4 illustrates these notions.

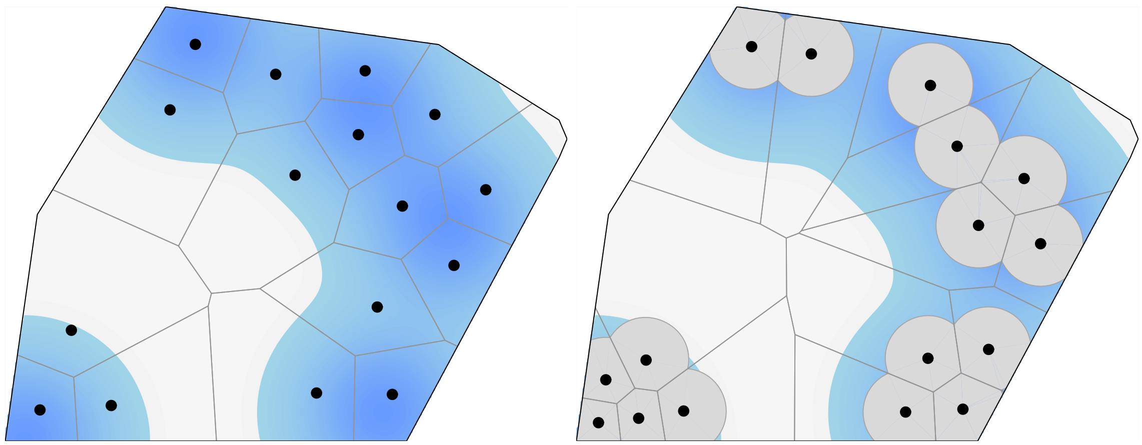

Fig. 4. Centroidal and $\frac{r}{2}$-centroidal Voronoi configurations. The density function ϕ is depicted by a contour plot. For each agent i, the set $V^{[i]} \cap \overline{B}(p_i, \frac{r}{2})$ is plotted in light gray.

Motivated by these observations, we define the following deployment task. For $r, \varepsilon \in \mathbb{R}_+$, define the ε-r-*deployment task* $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}} : Q^N \to \texttt{BooleSet}$ by

$$\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}(x) = \begin{cases} \texttt{true}, & \text{if } \left\| x^{[i]} - \text{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2})) \right\|_2 \leq \varepsilon, \text{ for all } i \in I\,, \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

$\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}(x) = \texttt{true}$ if and only if

$$\left\| x^{[i]} - \text{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) \right\|_2 \leq \varepsilon, \text{ for all } i \in I.$$

Roughly speaking, $\mathcal{T}_{\varepsilon\text{-}r\text{deplmnt}}$ is $\texttt{true}$ for those network configurations where each agent is sufficiently close to the centroid of an appropriate region $V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2})$.

B. Centroid law

To achieve the ε-r-deployment task discussed in Example IV-A, we define the *centroid* control and communication law $\mathcal{CC}_{\text{centrd}}$. This is a uniform, static, time-independent law studied in [11], [12]. Loosely speaking, the evolution of the network under the centroid control and communication law can be described as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the centroid of an appropriate region (the region is the intersection

between the agent's Voronoi cell and a closed ball centered at its position and of radius $\frac{r}{2}$), and (iii) it moves toward this centroid.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. We define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \text{Centroid}(\mathcal{X}) - x_{\text{smpld}}\,,$$

where $\mathcal{X} = Q \cap \overline{B}(x_{\text{smpld}}, \frac{r}{2}) \cap \big(\cap_{p \in \pi_L(y)} H_{x_{\text{smpld}},p}\big)$ and $H_{x_{\text{smpld}},p}$ is the half-space $\{q \in \mathbb{R}^d \mid \|q - x_{\text{smpld}}\|_2 \leq \|q-p\|_2\}$. One can show that Q^N is a positively-invariant set for this control law.

The following theorem on the centroid control and communication law summarizes the known results about the asymptotic properties and the novel results on the complexity of this law. In characterizing complexity, we assume $\text{diam}(Q)$ is independent of N, r and ε, and we do not calculate how the bounds depend on r. As for the circumcenter law, we provide complete time-complexity results for the case $d = 1$.

Theorem IV.1 (Time complexity of centroid law) *For $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, consider the network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ with initial conditions in Q. The following statements hold:*

(i) *for $d \in \mathbb{N}$, the law $\mathcal{CC}_{\text{centrd}}$ achieves the ε-r-deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$;*

(ii) *for $d = 1$ and $\phi = 1$, $\text{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(N^3 \log(N\varepsilon^{-1}))$.* •

Proof: Fact (i) is proved in [12] for $d \in \{1, 2\}$ and it is clear that the same proof technique can be generalized to any dimension. In what follows we sketch the proof of fact (ii). For $d = 1$, Q is a compact interval on $\mathbb{R}$, say $Q = [q_-, q_+]$.

We start with a brief discussion about connectivity. Note that in the r-limited Delaunay graph, two agents on the line that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Additionally we claim that, if agents i and j are neighbors at time instant ℓ, then $|\text{Centroid}(\mathcal{X}^{[i]}(\ell)) - \text{Centroid}(\mathcal{X}^{[j]}(\ell))| \leq r$. To see this, assume without loss of generality that $x^{[i]}(\ell) \leq x^{[j]}(\ell)$. Let us consider the case where the agents have neighbors on both sides (the other cases can be treated analogously). Let $x_-^{[i]}(\ell)$ (respectively, $x_+^{[j]}(\ell)$) denote the position of the neighbor of agent i to the left (respectively, of agent j to the right). Now, we have

$$\text{Centroid}(\mathcal{X}^{[i]}(\ell)) = \frac{1}{4}(x_-^{[i]}(\ell) + 2x^{[i]}(\ell) + x^{[j]}(\ell))\,, \quad \text{Centroid}(\mathcal{X}^{[j]}(\ell)) = \frac{1}{4}(x^{[i]}(\ell) + 2x^{[j]}(\ell) + x_+^{[j]}(\ell))\,.$$

Therefore, $|\operatorname{Centroid}(\mathcal{X}^{[i]}(\ell))-\operatorname{Centroid}(\mathcal{X}^{[j]}(\ell))| \leq \frac{1}{4}\big(|x_-^{[i]}(\ell)-x^{[i]}(\ell)|+2|x^{[i]}(\ell)-x^{[j]}(\ell)|+|x^{[j]}(\ell)-x_+^{[j]}(\ell)|\big) \leq r$. This implies that agents i and j are in the same connected component of the r-limited Delaunay graph at time instant $\ell+1$.

Next, let us consider the case that $E_{r\text{-LD}}(x_0)$ is connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. We distinguish three cases depending on the proximity of the leftmost and rightmost agents 1 and N, respectively, to the boundary of the environment: ($\mathfrak{a}$) both agents are within a distance $\frac{r}{2}$ of ∂Q; ($\mathfrak{b}$) none of the two is within a distance $\frac{r}{2}$ of ∂Q; and ($\mathfrak{c}$) only one of the agents is within a distance $\frac{r}{2}$ of ∂Q. Here is an important observation: from one time instant to the next one, the network configuration can fall into any of the cases described above. However, because of the discussion on connectivity, transitions can only occur from case ($\mathfrak{b}$) to either case ($\mathfrak{a}$) or ($\mathfrak{c}$); and from case ($\mathfrak{c}$) to case ($\mathfrak{a}$). As we show in the following, for each of these cases, the network evolution under $\mathcal{CC}_{\text{centrd}}$ can be described as a discrete-time linear dynamical system which respects agents' ordering.

Let us consider case ($\mathfrak{a}$). In this case, we have

$$x^{[1]}(\ell+1) = \frac{1}{4}(x^{[1]}(\ell)+x^{[2]}(\ell)) + \frac{1}{2}q_- , \quad x^{[2]}(\ell+1) = \frac{1}{4}(x^{[1]}(\ell)+2x^{[2]}(\ell)+x^{[3]}(\ell)) , \ \dots ,$$

$$\dots , \ x^{[N-1]}(\ell+1) = \frac{1}{4}(x^{[N-2]}(\ell)+2x^{[N-1]}(\ell)+x^{[N]}(\ell)) , \quad x^{[N]}(\ell+1) = \frac{1}{4}(x^{[N-1]}(\ell)+x^{[N]}(\ell)) + \frac{1}{2}q_+ .$$

Equivalently, we can write $x(\ell+1) = A_{\mathfrak{a}} \cdot x(\ell) + b_{\mathfrak{a}}$, where the $N \times N$-matrix $A_{\mathfrak{a}}$ and the vector $b_{\mathfrak{a}}$ are given by

$$A_{\mathfrak{a}} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} , \quad b_{\mathfrak{a}} = \begin{bmatrix} \frac{1}{2}q_- \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}q_+ \end{bmatrix} .$$

Note that the only equilibrium network configuration x_* respecting the ordering of the agents is given by

$$x_*^{[i]} = q_- + \frac{1}{2N}(1+2(i-1))(q_+ - q_-) , \quad i \in I ,$$

and note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case ($\mathfrak{a}$)). We can therefore write $(x(\ell) - x_*) = A_{\mathfrak{a}}(x(\ell-1) - x_*)$. Now, note that $A_{\mathfrak{a}} = \operatorname{ATrid}_N^-\left(\frac{1}{4}, \frac{1}{2}\right)$. Theorem A.4(ii) in [1] implies that

$\lim_{\ell\to+\infty}\big(x(\ell)-x_*\big)=\mathbf{0}$, and that the maximum time required for $\|x(\ell)-x_*\|_2\le\varepsilon\|x(0)-x_*\|_2$ (over all initial conditions $x(0)\in\mathbb{R}^N$) is $\Theta\big(N^2\log\varepsilon^{-1}\big)$. It is not obvious, but it can be verified, that the initial condition providing the lower bound in the time complexity estimate does indeed have the property of respecting the agents' ordering; this fact holds for all three cases ($\mathfrak{a}$), ($\mathfrak{b}$) and ($\mathfrak{c}$).

The case ($\mathfrak{b}$) can be treated in the same way. The network evolution takes now the form $x(\ell+1)=A_{\mathfrak{b}}\cdot x(\ell)+b_{\mathfrak{b}}$, where the $N\times N$-matrix $A_{\mathfrak{b}}$ and the vector $b_{\mathfrak{b}}$ are given by

$$A_{\mathfrak{b}}=\begin{bmatrix}\frac34 & \frac14 & 0 & \dots & \dots & 0\\ \frac14 & \frac12 & \frac14 & \dots & \dots & 0\\ 0 & \frac14 & \frac12 & \frac14 & \dots & 0\\ \vdots & & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & \dots & \frac14 & \frac12 & \frac14\\ 0 & \dots & \dots & 0 & \frac14 & \frac34\end{bmatrix},\quad b_{\mathfrak{b}}=\begin{bmatrix}-\frac14 r\\ 0\\ \vdots\\ 0\\ \frac14 r\end{bmatrix}.$$

In this case, a (non-unique) equilibrium network configuration respecting the ordering of the agents is of the form

$$x_*^{[i]}=ir-\frac{1+N}{2}r\,,\quad i\in I\,.$$

Note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case ($\mathfrak{b}$)). We can therefore write $(x(\ell)-x_*)=A_{\mathfrak{b}}(x(\ell-1)-x_*)$. Now, note that $A_{\mathfrak{b}}=\mathrm{ATrid}_N^+\left(\frac14,\frac12\right)$. We compute $x_{\text{ave}}=\frac1N\mathbf{1}^T(x_0-x_*)=\frac1N\mathbf{1}^Tx_0$. With this calculation, Theorem A.4(i) in [1] implies that $\lim_{\ell\to+\infty}\big(x(\ell)-x_*-x_{\text{ave}}\mathbf{1}\big)=\mathbf{0}$, and that the maximum time required for $\|x(\ell)-x_*-x_{\text{ave}}\mathbf{1}\|_2\le\varepsilon\|x(0)-x_*-x_{\text{ave}}\mathbf{1}\|_2$ (over all initial conditions $x(0)\in\mathbb{R}^N$) is $\Theta\big(N^2\log\varepsilon^{-1}\big)$.

Case ($\mathfrak{c}$) needs to be handled differently. Without loss of generality, assume that agent 1 is within distance $\frac{r}{2}$ of ∂Q and agent N is not (the other case is treated analogously). Then, the network evolution takes now the form

$x(\ell+1) = A_{\mathfrak{c}} \cdot x(\ell) + b_{\mathfrak{c}}$, where the $N \times N$-matrix $A_{\mathfrak{c}}$ and the vector $b_{\mathfrak{c}}$ are given by

$$A_{\mathfrak{c}} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \quad b_{\mathfrak{c}} = \begin{bmatrix} \frac{1}{2}q_- \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}r \end{bmatrix}.$$

Note that the only equilibrium network configuration x_* respecting the ordering of the agents is given by

$$x_*^{[i]} = q_- + \frac{1}{2}(2i-1)r\,, \quad i \in I\,,$$

and note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case ($\mathfrak{c}$)). In order to analyze $A_{\mathfrak{c}}$, we recast the N-dimensional discrete-time dynamical system as a $2N$-dimensional one. To do this, we define a $2N$-dimensional vector y by

$$y^{[i]} = x^{[i]}, i \in I, \quad \text{and} \quad y^{[N+i]} = x^{[N-i+1]}, i \in I, \tag{8}$$

Now, one can see that the network evolution can be alternatively described in the variables $(y^{[1]}, \dots, y^{[2N]})$ as a linear dynamical system determined by the $2N \times 2N$ matrix $\mathrm{ATrid}^-_{2N}(\frac{1}{4}, \frac{1}{2})$. Using analogous arguments to the ones used before and exploiting the chain of equalities (8), we can characterize the eigenvalues and eigenvectors of $\mathrm{Trid}_{2N-1}(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, and infer that, even for case ($\mathfrak{c}$), the maximum time required for $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$.

In summary, for all three cases ($\mathfrak{a}$), ($\mathfrak{b}$) and ($\mathfrak{c}$), our calculations show that, in time $O\big(N^2 \log \varepsilon^{-1}\big)$, the error 2-norm satisfies the contraction inequality $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$. We convert this inequality on 2-norms into an appropriate inequality on ∞-norms as follows. Note that $\|x(0) - x_*\|_\infty = \max_{i \in I} |x^{[i]}(0) - x_*^{[i]}| \leq (q_+ - q_-)$. For ℓ of order $N^2 \log \eta^{-1}$, we have

$$\|x(\ell) - x_*\|_\infty \leq \|x(\ell) - x_*\|_2 \leq \eta \|x(0) - x_*\|_2 \leq \eta\sqrt{N} \|x(0) - x_*\|_\infty \leq \eta\sqrt{N}(q_+ - q_-).$$

This means that ε-r-deployment is achieved for $\eta\sqrt{N}(q_+ - q_-) = \varepsilon$, that is, in time $O(N^2 \log \eta^{-1}) = O(N^2 \log(N\varepsilon^{-1}))$.

Up to here we have proved that, if the graph $(I, E_{r\text{-LD}}(x_0))$ is connected, then $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(N^2 \log(N\varepsilon^{-1}))$. If $(I, E_{r\text{-LD}}(x_0))$ is not connected, note that along the network evolution there can only be a finite number of time instants, at most $N-1$ where a merging of two connected components occurs. Therefore, the time complexity is at most $O(N^3 \log(N\varepsilon^{-1}))$. ■

V. CONCLUSIONS

Building on the framework proposed in the companion paper [1] to model and analyze robotic networks, we have formalized various motion coordination algorithms: the move-toward-average and the circumcenter laws, achieving the rendezvous task, and the centroid law, achieving the deployment task. We have computed the time complexity of these algorithms, providing upper and lower bounds as the number of agents tends to infinity. To obtain these complexity estimates, we have developed some novel analysis methods involving linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices. These results demonstrate the usefulness of the proposed formal model. We hope that they will help assess the complex trade-offs between computation, communication and motion control in robotic networks.

A number of research avenues look now promising and exciting. In this paper, our analysis results essentially consist of a time-complexity analysis of some basic algorithms, but many more open algorithmic questions remain unresolved including (i) analysis of the communication complexity for unidirectional and omnidirectional models of communication; (ii) analysis of other known algorithms for flocking, cohesion, formation, motion planning and a long etcetera; and (iii) complexity analysis results for coordination tasks, as opposed to for algorithms.

ACKNOWLEDGMENTS

This material is based upon work supported in part by ONR YIP Award N00014-03-1-0512, NSF SENSORS Award IIS-0330008, DARPA/AFOSR MURI Award F49620-02-1-0325, and NSF CAREER Award CCR-0133869. Sonia Martínez's work was supported in part by a Fulbright Postdoctoral Fellowship from the Spanish Ministry of Education and Science.

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APPENDIX I

BASIC GEOMETRIC NOTIONS

Here we have gathered various geometric concepts used throughout the paper. Let $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be compact. The *circumcenter of* S, denoted by $\operatorname{Circum}(S)$, is the center of the smallest-radius sphere in $\mathbb{R}^d$ enclosing S. Given an integrable function $\phi\colon S \to \mathbb{R}_+$, the *mass* of S is $\operatorname{Mass}(S) = \int_S \phi(q)dq$, and the *centroid* of S is

$$\operatorname{Centroid}(S) = \frac{1}{\operatorname{Mass}(S)} \int_S q\phi(q)dq \,.$$

A *partition* of S is a collection of subsets of S with disjoint interiors and whose union is S. Given a set of N distinct points $\mathcal{P} = \{p_i\}_{i\in\{1,\dots,N\}}$ in S, the *Voronoi partition of* S generated by $\mathcal{P}$ (with respect to the Euclidean norm) is the collection of sets $\{V_i(\mathcal{P})\}_{i\in\{1,\dots,N\}}$ defined by $V_i(\mathcal{P}) = \{q \in S \mid \|q - p_i\|_2 \le \|q - p_j\|_2 \,, \text{ for all } p_j \in \mathcal{P}\}$. We usually refer to $V_i(\mathcal{P})$ as V_i. For a detailed treatment of Voronoi partitions we refer to [24], [23].

For $I = \{1, \dots, N\}$ and $S \subset \mathbb{R}^d$, a *proximity edge map* is a map of the form $E\colon S^N \to 2^{I\times I\setminus \operatorname{diag}(I\times I)}$. For $r \in \mathbb{R}_+$, we define the r-disk proximity edge map $E_{r\text{-disk}}\colon (\mathbb{R}^d)^N \to 2^{I\times I}$ and the r-limited Delaunay proximity edge map $E_{r\text{-LD}}\colon (\mathbb{R}^d)^N \to 2^{I\times I}$ as follows. An edge $(i,j) \in I \times I \setminus \operatorname{diag}(I\times I)$ belongs to $E_{r\text{-disk}}(x_1, \dots, x_N)$ if and only if $\|x_i - x_j\|_2 \le r$. An edge $(i,j) \in I \times I \setminus \operatorname{diag}(I\times I)$ belongs to $E_{r\text{-LD}}(x_1, \dots, x_N)$ if and only if

$$\big(V_i \cap \overline{B}(x_i, \tfrac{r}{2})\big) \cap \big(V_j \cap \overline{B}(x_j, \tfrac{r}{2})\big) \neq \emptyset,$$

where $\{V_1, \dots, V_N\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x_1, \dots, x_N\}$. Illustrations of these concepts are given in Fig. 5.

As proved in [12], the r-limited Delaunay graph and the r-disk graph have the same connected components. Additionally, the r-limited Delaunay graph is “computable” on the r-disk graph in the following sense: any node in the network can compute the set of its neighbors in the r-limited Delaunay graph if it is given the set of its neighbors

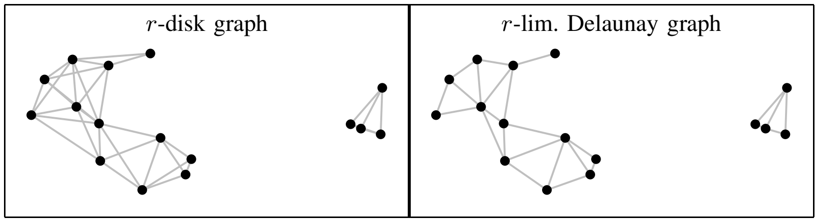

Fig. 5. The r-disk and r-limited Delaunay graphs in $\mathbb{R}^2$.

in the r-disk graph. This implies that any control and communication law for a network with communication graph $E_{r\text{-LD}}$ can be implemented on a analogous network with communication graph $E_{r\text{-disk}}$.