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Distributed representation of spatial fields through an adaptive interpolation scheme

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Abstract— **We present a procedure to adapt an interpolation scheme to represent spatial fields as they are measured by a mobile sensor network. The procedure incorporates new sensor (synchronous) measurements in a similar fashion to a Kalman filter-like recursion. Using ideas from distributed consensus algorithms, we show how the scheme corresponding to nearest-neighbor interpolations admits decentralization over a directed proximity graph related to the Delaunay graph.**

I. Introduction

An intensive research activity is being directed to the development of coordination algorithms that allow the practical use of multi-vehicle sensor networks. Examples of such systems used in exploration and scientific ventures include multi-buoy systems, coordinated gliders for oceanographic research, and unmanned aerial vehicles (UAVs) for e.g., atmospheric observation.

Typically, these sensor networks are required to communicate with a central data fusion station that gathers all the measurement information to produce an approximation of the fields of interest. Although it is useful to have all the information at one place so that users can have access to it, the possibility of placing part of the estimation load on the vehicles themselves would make their reaction to events more efficient. In order to make this possible, the identification of suitable methods for cooperative estimation and conditions for their distributed computation should be investigated.

Literature review. The investigation of the requirements for information processing in a decentralized setup dates back to the '80; see e.g., [1], and is related to the area of sensor fusion. The synthesis of distributed coordination algorithms for multi-agent and sensor systems is the subject of current research. In particular, consensus algorithms [2], [3] have been widely analyzed and proposed for sensor fusion [4], and as a way to decentralize Kalman filters [5]. The devise of optimal sensor placement or motion coordination plans have been recently addressed to improve estimation procedures such as Kalman filters for target tracking [6], or the optimal sampling of spatial fields [7]. The assumption of fixed communication topologies or the existence of a central station that is able to fuse information and communication with all vehicles is a restriction mostly considered. A related paper to the present work is [8], which investigates user information retrieval protocols from a static sensor network based on a nearest-neighbor partition of the space. However, [8] leaves the problem of sensor data fusion unaddressed. Another related issue that we do not treat here is that of energy efficient estimation or detection in sensor networks.

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Statement of contributions. We analyze a procedure to modify (nearest-neighbor) interpolation schemes to represent spatial fields by a multi-vehicle system. The interpolation provides an estimate of the field, which is refined via a Kalman filter-like recursion as new measurements are collected. We derive the expression of the optimal gains of the filter and obtain conditions under which the scheme admits decentralization in the nearest-neighbor interpolation case. One of such conditions requires agents to reach consensus on the values of the optimal gains. The required inter-vehicle communication graph should also contain a newly-identified proximity graph that is related to the Delaunay graph.

Organization of the paper. The paper is organized as follows. In Section II, we state the main problem scenario and review basic concepts on Voronoi partitions and spatial interpolation methods. In Section III, we propose how to adapt a (nearest-neighbor) interpolation scheme by means of a Kalman filter-like procedure. This includes the derivation of the optimal gains that should be computed at each step of the discrete-time algorithm. Finally, Section IV presents the requirements for distributing the adaptive interpolation method.

II. Problem statement and preliminaries

Here we state the general problem scenario with given assumptions, and introduce basic preliminaries on Voronoi partitions and spatial interpolation methods.

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A. Motivating Problem and Assumptions

Let $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$) denote the positive real numbers including 0 (resp. the strictly positive real numbers) and let $\{p_1, \dots, p_n\}$ denote the positions of n vehicles moving on a compact and convex region of the space $Q \subseteq \mathbb{R}^3$. We assume each vehicle $i \in \{1, \dots, n\}$ is endowed with physico-chemical sensors and is able to take point measurements z_i of certain scalar field $\phi : \mathbb{R} \times Q \to \mathbb{R}_{\geq 0}$. For example ϕ might represent an environmental substance such as salinity concentration in the sea or aerosol pollutant in the atmosphere.

For simplicity we will consider here that ϕ is *static*; i.e., $\phi : Q \to \mathbb{R}_{\geq 0}$. This is a reasonable assumption when measuring substance concentrations such as aerosol fields (which do not change considerably with time under mild weather conditions). We will also assume that the measurements z_i, $i \in \{1, \dots, n\}$, are affected by a spatially and temporally uncorrelated white noise. That is, $z_i(t) = \phi(p_i(t)) + \epsilon_i(t)$, with $\epsilon_i(t) \sim \mathcal{N}(0, \sigma)$, $\forall t \geq 0$, $i \in \{1, \dots, n\}$, and $E[\epsilon_i(t)\epsilon_j(s)] = 0$ if either $i \neq j$, or $t \neq s$ $\forall\, t, s \geq 0$, and $i, j \in \{1, \dots, n\}$.

B. Voronoi Partitions and the Delaunay Graph

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^3$. Recall that the (Euclidean) Voronoi partition of $Q \subseteq \mathbb{R}^3$ associated with a set of n distinct points $\mathcal{P} = \{p_1, \dots, p_n\} \subseteq Q$ is a collection of sets $\mathscr{V}(\mathcal{P}) = \{V_i(\mathcal{P})\}_{i=1}^n$ such that $\cup_{i=1}^n V_i(\mathcal{P}) = Q$, and $V_i(\mathcal{P})$ is the region defined as:

$$V_i(\mathcal{P}) = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\| \text{ for all } j \neq i\}\,,$$

for all $i \in \{1, \dots, n\}$. We will usually refer to $V_i(\mathcal{P})$ as V_i. It is easy to see that p_i belongs to its Voronoi region V_i, $i \in \{1, \dots, n\}$. We say that p_j is a *Voronoi neighbor* of p_i if and only if V_i and V_j have a non-empty intersection. We denote the set of Voronoi neighbors of i as $\mathcal{N}_i$. This relationship gives rise to the undirected *Delaunay graph*, $\mathcal{G}_{\mathrm{D}} = (\mathcal{V}, \mathcal{E})$, defined over the set of vertices $\mathcal{V} = \{1, \dots, n\}$ and edge set $\mathcal{E} = \{(i,j) \mid i \in \mathcal{N}_j\,,\ \ j \in \{1, \dots, n\}\}$. For a detailed treatment of Voronoi partitions and the Delaunay graph we refer the reader to [9].

C. Spatial Interpolation Methods

There are several methods available to predict multivariate fields $\phi : Q \to \mathbb{R}_{\geq 0}$ from scattered data.

In the absence of measurement noise, the general formulation of a spatial interpolation problem is the following: Given the n values of the studied phenomenon, $z_i = \phi(p_i)$, $i \in \{1, \dots, n\}$, measured at discrete points $\{p_1, \dots, p_n\}$, find a function $\Phi : Q \to \mathbb{R}_{\geq 0}$ such that $\Phi(p_i) = z_i$, for all $i \in \{1, \dots, n\}$.

An interpolant Φ is called *global* (resp. *local*), when the value of Φ at any point $q \in Q$ depends on *all* the data values (resp. only on data values at *"nearby"* points). Global interpolants are affected by the addition or deletion of data values and by changes in the location of data sites, while local interpolants are only affected at a vicinity of the changes. The required scalability properties of distributed systems and their decentralized nature make local interpolants more readily adaptable for groups of multiple vehicles.

Some of the most widely used local interpolation methods include Inverse Distance Weighted Interpolation (IDW), Nearest and Natural Neighbor Interpolations (NN), and interpolations based on a Triangulated Irregular Networks (TIN) [10], [9], [11]. The simplest interpolation of a function over Q is given by the nearest neighbor rule:

$$\phi(q) = z_i\,, \qquad \|q - p_i\| < \|q - p_j\|, \quad j \neq i\,.$$

The resulting function is discontinuous at the boundaries of the Voronoi regions $V_i(\mathcal{P})$ associated with the location of sites $\mathcal{P}$. An extension of this method is the Natural Neighbors interpolation method, defined as follows. Given a point $q \in Q$ and a set of locations $\mathcal{P}$, compute $\mathscr{V}(\mathcal{P} \cup \{q\})$. The value $\Phi(q)$ is a linear combination $\Phi(q) = \sum_{i \in \mathcal{N}(q)} w_i z_i$, where $\mathcal{N}(q)$ denotes the set of neighbors of q in the Delaunay graph associated with $\mathscr{V}(\mathcal{P} \cup \{q\})$ and $\{w_i\}_{i \in \mathcal{N}(q)}$ are a priori defined weights.

Although the NN approaches do not give rise to continuous representations, they are computationally very fast and can be easily extended to any bounded set of any dimension. In comparison, the TIN approaches require the computation of a set of generalized tetrahedra in $\mathbb{R}^n$, which can lead to complications when defining partitions of compact domains. A solution to deal with this problem, see [9], requires the placement of many nodes along the boundary of Q.

Here we will investigate a means of refining the simpler nearest-neighbor interpolation, and leave for future works the investigation of other local and global interpolation methods.

III. Centralized interpolation filter

This section describes the centralized interpolation filter scheme that makes use of the NN interpolation rule and is refined through a Kalman filter-like procedure.

From now on assume that there is a scheduling time sequence $\mathbb{T} = \{t_\ell \mid \ell \in \mathbb{N}\}$ known by each agent that synchronizes the taking of the n independent measurements $z_i(t_\ell)$, $i \in \{1, \dots, n\}$, $t_\ell \in \mathbb{T}$. This is a reasonable assumption for static fields, where waiting time periods for all vehicles can be established.

Let $\mathcal{P}_\ell = \{p_1^\ell, \dots, p_n^\ell\}$ denote the positions of the n vehicles at time $t_\ell \in \mathbb{T}$ and let $\mathscr{V}(\mathcal{P}_\ell) = \{V_1^\ell, \dots, V_n^\ell\}$ denote the associated Voronoi partition of Q. We define the class of functions $\overline{\mathscr{C}} = \{\overline{\psi} : \mathbb{R} \times Q \to \mathbb{R}_{>0} \,|\, \forall t \in \mathbb{R}, \overline{\psi}(t, \cdot)$ is piecewise constant as a function of $Q\}$ and let $\mathscr{C} = \{\psi : \mathbb{R} \times Q \to \mathbb{R} \,|\, \exists \overline{\psi} \in \overline{\mathscr{C}}$ such that $\psi(t, \cdot) \sim \mathcal{N}(\overline{\psi}(t, \cdot), \sigma)$ and $E[\psi(t, p)\psi(s, q)] = 0$ for $t \neq s$ or $p \neq q\}$. Now we define an (nearest-neighbor) *observation operator*, $\mathcal{Q} : \mathbb{T} \times Q^n \times \overline{\mathscr{C}} \to \mathscr{C}$, for a given time schedule $\mathbb{T}$, tuples of positions $\mathcal{P}_\ell \in Q^n$, $\ell \in \mathbb{N}$, and spatio-temporal fields $\overline{\psi} \in \overline{\mathscr{C}}$. That is, $\mathcal{Q}(t_\ell, \mathcal{P}_\ell, \overline{\psi}) \in \mathscr{C}$ is a new static spatial field defined as:

$$\mathcal{Q}(t_\ell, \mathcal{P}_\ell, \overline{\psi})(q) = \sum_{i=1}^{n} (\overline{\psi}(t_\ell, p_i^\ell) + \epsilon(t_\ell, p_i^\ell)) \cdot 1_{V_i^\ell}(q) \,,$$

for all $q \in Q$. Here $\overline{\psi}(t_\ell, p_i^\ell) + \epsilon(t_\ell, p_i^\ell)$ is the measurement of $\overline{\psi}$ that sensor at p_i^ℓ takes, where $\epsilon : \mathbb{R} \times Q \to \mathbb{R}$ is a white noise such that $\epsilon(t, p) \sim \mathcal{N}(0, \sigma)$, $E[\epsilon(t, p)\epsilon(s, q)] = 0$ for $t \neq s$ or $p \neq q$. The function $1_{V_i^\ell}(q)$ is the indicator function $1_{V_i^\ell}(q) = 1$, if $q \in V_i^\ell$, otherwise $1_{V_i^\ell}(q) = 0$. In other words, $\mathcal{Q}$ provides a *snapshot* of a given $\overline{\psi}$ according to measurements at vehicle sites $\mathcal{P}_\ell$ at time $t_\ell \in \mathbb{T}$. For simplicity we will use the notation $\mathcal{Q}_\ell \psi \equiv \mathcal{Q}(t_\ell, \mathcal{P}_\ell, \psi)$, whenever it is clear that the sites $\mathcal{P}_\ell$ correspond to the vehicles' positions at time $t_\ell \in \mathbb{T}$ and $\epsilon_i^\ell = \epsilon(t_\ell, p_i^\ell)$, for $\ell \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. Associated with it, one can define an *averaged observation operator* as $\overline{\mathcal{Q}} : \mathbb{T} \times Q^n \times \overline{\mathscr{C}} \to \overline{\mathscr{C}}$ such that $\overline{\mathcal{Q}}_\ell \overline{\psi} \equiv \overline{\mathcal{Q}}(t_\ell, \mathcal{P}, \overline{\psi}) = \sum_{i=1}^n \overline{\psi}(t_\ell, p_i) 1_{V_i^\ell}(q)$.

Let $\phi : Q \to \mathbb{R}_{\geq 0}$ be the static field we would like to represent. After taking a collective measurement of ϕ at time $t_0 = 0$, the first estimate can be taken to be $\overline{\phi}_0 = \overline{\mathcal{Q}}_0 \, \phi = E[\mathcal{Q}_0 \, \phi]$. As new measurements are taken, we use an update rule inspired by a Kalman filter to refine the interpolation. The convex combination:

$$\phi_\ell = \phi_{\ell-1} + W_\ell(\mathcal{Q}_\ell \phi - \mathcal{Q}_\ell \phi_{\ell-1}), \quad \ell \geq 1 \,,$$

yields an estimated value $\overline{\phi}_\ell = E[\phi_\ell]$ of the field ϕ at time t_ℓ. By an induction argument, one can see that:

$$\overline{\phi}_\ell = \overline{\phi}_{\ell-1} + W_\ell(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}) \,, \quad \ell \geq 1. \quad (1)$$

As in a Kalman filter,W_ℓ plays the role of the gain at time $t_\ell \in \mathbb{T}$. The combination (1) is a weighted sum of the predicted value of the field, $\overline{\phi}_{\ell-1}$, and the *measurement innovation*, $\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}$, where $\mathcal{Q}_\ell \phi$ is the new observation of ϕ and $\mathcal{Q}_\ell \overline{\phi}_\ell$ is the predicted measurement. The update rule (1) is understood as a point-wise equality for all $q \in Q$.

Given $\psi \in \overline{\mathscr{C}}$ and an approximation $E[\hat{\psi}] \approx \psi$, with $\hat{\psi} \in \mathscr{C}$, we define the minimum square error (MSE) as:

$$\mathrm{MSE}(\psi, \hat{\psi}) = \int_Q E[(\psi(q) - \hat{\psi}(q))^2] \, dq \,.$$

In the following, we obtain an expression for $\mathrm{MSE}(\phi, \phi_\ell)$, $\ell \in \mathbb{N}$, in order to find the optimal value of the gains which minimize this error. For simplicity we will use the notation $\mathrm{MSE}(\phi, \phi_\ell) \equiv \mathrm{MSE}_\ell$, $\ell \in \mathbb{N}$.

Lemma 1: The following equalities hold for all $\ell \in \mathbb{N}$ and $q, p \in Q$:

$$E[\phi_{\ell-1}(q) \, \mathcal{Q}_\ell \phi(p)] = \overline{\phi}_{\ell-1}(q) \, \overline{\mathcal{Q}}_\ell \phi(p) \,,$$
$$E[\mathcal{Q}_\ell \phi(q) \, \mathcal{Q}_\ell \phi(p)] = \overline{\mathcal{Q}}_\ell \phi(q) \, \overline{\mathcal{Q}}_\ell \phi(p) + \sigma^2 \,,$$
$$E[\mathcal{Q}_\ell \phi(q) \, \mathcal{Q}_\ell \phi_{\ell-1}(p)] = \overline{\mathcal{Q}}_\ell \phi(q) \, \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}(p) + \sigma^2 \,.$$

Using these formulas, one can obtain the expression:

$$E[\phi_\ell(q) \, \phi_\ell(p)] = \overline{\phi_\ell}(q) \, \overline{\phi}_\ell(p) + \sigma^2 \Pi_{s=1}^{\ell} (1 - W_s)^2. \quad (2)$$

Proof: See the proof in the extended report [12]. ■

Lemma 2: The following equalities hold for all $\ell \in \mathbb{N}$:

$$E[(\mathcal{Q}_\ell \phi)^2] = (\overline{\mathcal{Q}}_\ell \phi)^2 + \sigma^2 \,,$$
$$E[(\mathcal{Q}_\ell \phi_{\ell-1})^2] = (\overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})^2 + \sigma^2 \left(1 + \Pi_{s=1}^{\ell-1} (1 - W_s)^2\right) \,,$$
$$E[\phi \, \mathcal{Q}_\ell \phi] = \phi \, \overline{\mathcal{Q}}_\ell \phi \,,$$
$$E[\phi \, \mathcal{Q}_\ell \phi_{\ell-1}] = \phi \, \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1} \,,$$
$$E[\phi_{\ell-1} \, \mathcal{Q}_\ell \phi_{\ell-1}] = \overline{\phi}_{\ell-1} \, \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1} + \sigma^2 \Pi_{s=1}^{\ell-1} (1 - W_s)^2 \,.$$

Using these formulas and Lemma 1, it is possible to obtain the recursive expression for $\ell \geq 1$:

$$\begin{aligned} \mathrm{MSE}_\ell &= \mathrm{MSE}_{\ell-1} + \\ & W_\ell^2 \Big(\int_Q (\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})^2 \, dq + \sigma^2 \Pi_{s=1}^{\ell-1} (1 - W_s)^2 M_Q \Big) \\ & - 2W_\ell \Big(\int_Q (\phi - \overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}) \, dq \\ & + \sigma^2 \Pi_{s=1}^{\ell-1} (1 - W_s)^2 M_Q \Big) \,, \quad (3) \end{aligned}$$

where M_Q is the volume of Q, $M_Q = \int_Q dq$.

Proof: See the proof in the report [12]. ■

Theorem 3: For given values W_s, $s \in \{1, \dots, \ell - 1\}$, the optimal gain W_ℓ^* that guarantees $\mathrm{MSE}_\ell \leq \mathrm{MSE}_{\ell-1}$, for all $\ell \in \mathbb{N}$, and is a local minimum of MSE_ℓ is:

$$W_\ell^* = \frac{\sum_{i=1}^n (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell)) \int_{V_i^\ell} (\phi(q) - \overline{\phi}_{\ell-1}(q)) dq + C}{\sum_{i=1}^n (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell))^2 M_{V_i^\ell} + C}$$

where $C = \sigma^2\Pi_{s=1}^{\ell-1}(1-W_s)^2 M_Q$ and $M_{V_i^\ell}$ is the volume or *mass* of the Voronoi region V_i^ℓ; i.e., $M_{V_i^\ell} = \int_{V_i^\ell} dq$, $i \in \{1,\dots,n\}$.

Proof: Taking the partial derivative of MSE_ℓ with respect to W_ℓ in (3) and equating this to zero, we obtain:

$$W_\ell\Big(\int_Q(\overline{\mathcal{Q}}_\ell\phi-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1})^2\,dq+\sigma^2\Pi_{s=1}^{\ell-1}(1-W_s)^2M_Q\Big) - \int_Q(\phi-\overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell\phi-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1})\,dq - \sigma^2\Pi_{s=1}^{\ell-1}(1-W_s)^2M_Q = \frac{1}{2}\frac{\partial \mathrm{MSE}_\ell}{\partial W_\ell} = 0\,.$$

The critical value of the gain, W_ℓ^*, is thus given by:

$$\frac{\int_Q(\phi-\overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell\phi-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1})dq+C}{\int_Q(\overline{\mathcal{Q}}_\ell\phi-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1})^2\,dq+C}\,,$$

which satisfies $\frac{\partial^2 \mathrm{MSE}_\ell}{\partial W_\ell^2}_{|W_\ell^*} > 0$; thus W_ℓ^* is a local minimum. Now, using that $\mathscr{V}(\mathcal{P}_\ell)$ is a partition of Q:

$$\begin{aligned}
&\int_Q(\overline{\mathcal{Q}}_\ell\phi(q)-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1}(q))^2\,dq \\
&=\sum_{i=1}^n\int_{V_i^\ell}(\overline{\mathcal{Q}}_\ell\phi(q)-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1}(q))^2\cdot 1_{V_i^\ell}(q)\,dq \\
&=\sum_{i=1}^n\int_{V_i^\ell}((\overline{\mathcal{Q}}_\ell\phi(q)-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1}(q))\cdot 1_{V_i^\ell}(q))^2\,dq \\
&=\sum_{i=1}^n\int_{V_i^\ell}((\sum_{j=1}^n(\phi(p_j^\ell)-\overline{\phi}_{\ell-1}(p_j^\ell))\cdot 1_{V_j^\ell}(q))1_{V_i^\ell}(q))^2dq \\
&=\sum_{i=1}^n\int_{V_i^\ell}(\phi(p_i^\ell)-\overline{\phi}_{\ell-1}(p_i^\ell))^2\cdot 1_{V_i^\ell}(q)dq \\
&=\sum_{i=1}^n(\phi(p_i^\ell)-\overline{\phi}_{\ell-1}(p_i^\ell))^2M_{V_i^\ell}\,,
\end{aligned}$$

where we have used the fact that $1_{V_i^\ell}(q)\cdot 1_{V_j^\ell}(q)$ is identically zero for all $i \neq j$ except for a set of measure zero, $\partial V_i^\ell \cap \partial V_j^\ell$. A similar computation leads to:

$$\int_Q(\phi-\overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell\phi-\overline{\mathcal{Q}}_\ell\overline{\phi}_{\ell-1}) = \sum_{i=1}^n(\phi(p_i^\ell)-\overline{\phi}_{\ell-1}(p_i^\ell))\int_{V_i^\ell}(\phi(q)-\overline{\phi}_{\ell-1}(q))dq\,.$$

Thus the claimed expression for W_ℓ^* is obtained. Finally, to see that $\mathrm{MSE}_\ell \leq \mathrm{MSE}_{\ell-1}$ with W_ℓ^*, we substitute $0 = W_\ell^*\cdot\frac{1}{2}\frac{\partial\,\mathrm{MSE}_\ell}{\partial W_\ell}_{|W_\ell^*}$ into (3) to obtain:

$$\begin{aligned}
&\mathrm{MSE}_\ell = \\
&\mathrm{MSE}_{\ell-1} - W_\ell^*\Big(\int_Q(\phi-\overline{\phi}_{\ell-1})(\mathcal{Q}_\ell\phi-\mathcal{Q}_\ell\overline{\phi}_{\ell-1})+C\Big) \\
&= \mathrm{MSE}_{\ell-1} - \frac{(\int_Q(\phi-\overline{\phi}_{\ell-1})(\mathcal{Q}_\ell\phi-\mathcal{Q}_\ell\overline{\phi}_{\ell-1})\,dq+C)^2}{\int_Q(\mathcal{Q}_\ell\phi-\mathcal{Q}_\ell\overline{\phi}_{\ell-1})^2dq+C} \\
&\qquad \leq \mathrm{MSE}_{\ell-1}\,, \qquad \forall \ell \in \mathbb{N}\,,
\end{aligned}$$

since the factor we subtract to $\mathrm{MSE}_{\ell-1}$ is positive. ■

Remark 4: The computation of the optimal gain, W_ℓ^*, requires precise knowledge about the value of the integral of ϕ over the Voronoi regions V_i^ℓ, $i \in \{1,\dots,n\}$ (integral in the numerator). With limited information about ϕ, each vehicle can only compute this value *approximately* through e.g., quadrature rules [13].

Suppose ϕ is Lipschitz over Q and let $\Omega \subseteq Q$. A quadrature rule for the computation of $\int_\Omega \phi(q)dq$ is defined as:

$$\int_\Omega \phi(q)dq \approx \sum_{k=1}^m \phi(q_k)\cdot M_{A_k} \tag{4}$$

where $q_i \in \Omega$ and $\{A_k\}_{k=1}^m$ is a partition of Ω associated with $(q_1,\dots,q_m) \in \Omega^m$. The subtraction of both terms in (4) can be bounded as follows:

$$\begin{aligned}
&\Big|\int_\Omega \phi(q)dq - \sum_{k=1}^m \phi(q_k)\cdot M_{A_k}\Big| \\
&\leq \sum_{k=1}^m\int_{A_k}|\phi(q)-\phi(q_i)|dq \leq L\sum_{k=1}^m\int_{A_k}|q-q_i|dq,
\end{aligned}$$

where L is the Lipschitz constant of ϕ. When $k \to \infty$, $M_{A_k} \approx 0$, the above is a good approximation. For a finite number of measurements in Ω, it can be proven that the quadrature is minimized for A_k Voronoi regions and $q_k \in A_k$ being at the *centroids* of these regions; i.e., $q_k = C_{A_k}$, with $M_{A_k}\cdot C_{A_k} = \int_{A_k} q dq$, for all $k \in \{1,\dots,m\}$. Based on this, each vehicle could take the simple approximation $\int_{V_i^\ell}\phi(q)dq \approx \phi(C_{V_i^\ell})M_{V_i^\ell}$ (optimal), or $\int_{V_i^\ell}\phi(q)dq \approx \phi(p_i^\ell)M_{V_i^\ell}$ (non-optimal); see [13]. The gain obtained in this way, $\widehat{W}_\ell$ is an approximation of W_ℓ^*. As more measurements of ϕ are stored by vehicles (e.g., possibly taken along a path from p_i^ℓ to $C_{V_i^\ell}$) the approximation will improve and we will have $\widehat{W}_\ell \to W_\ell^*$ as $\ell \to \infty$. From now on, we will assume that this type of approximation is taking place.•

IV. DISTRIBUTED INTERPOLATION FILTER

The decentralization of the previous interpolation filter can only be done if vehicles can compute the approximated gains W_ℓ^*, $\ell \in \mathbb{N}$, in a distributed manner. On the other hand, it is not necessary that each vehicle maintains a global representation of ϕ, but just a local one over its region V_i^ℓ, $i \in \{1,\dots,n\}$. In this section, we derive necessary conditions on vehicle communication that allow the distributed implementation of the filter.

Our first assumption is that each robotic agent or vehicle will be able to take measurements and communicate with others according to a refined time scheduling sequence $\overline{\mathbb{T}}$ defined as follows. Let $\mathbb{T} = \{t_\ell\}_{\ell\in\mathbb{N}}$ be a time scheduling sequence for synchronous vehicle measurement. Consider $\{s_\ell^m\}_{m=1}^{K_\ell}$ for each $\ell \in \mathbb{N}$ and $K_\ell \in \mathbb{N}$ such that:

$$[t_\ell, t_{\ell+1}] = [t_\ell, t_\ell + s_\ell^1] \cup [t_\ell + s_\ell^2] \cup \dots \cup [t_\ell + s_\ell^{K_\ell - 1}, t_{\ell+1}]\,,$$

where $t_{\ell+1} = t_\ell + s_\ell^{K_\ell}$. In this way, $\overline{\mathbb{T}} = \{t_\ell + s_\ell^m \mid \ell \in \mathbb{N},\ m \in \{1,\dots,K_\ell\}\}$. We use the notation $t_{\ell m} = t_\ell + s_\ell^m \in \overline{\mathbb{T}}$ from now on.

At each time $t_{\ell m} \in \overline{\mathbb{T}}$, we assume that an undirected communication graph is established. That is, $\mathcal{G}(t_{\ell m}) = (\{1,\dots,n\}, \mathcal{E}(t_{\ell m}))$, with $\mathcal{E}(t_{\ell m})$ the set of edges in the graph at time $t_{\ell m}$. That is, i and j can exchange a message at time $t_{\ell m} \in \overline{\mathbb{T}}$ if and only if $(i,j) \in \mathcal{E}(t_{\ell m})$. The idea is that the distributed computation of the gain $W_{\ell+1}^*$ will be done through communication rounds at the times $t_{\ell 1},\dots,t_{\ell K_\ell}$, as follows.

Suppose that $\phi_0 \equiv 0$ and the following assumptions on agent $i \in \{1,\dots,n\}$ hold for all time t_ℓ:

(i) Each agent i has knowledge of V_i^ℓ and $\overline{\phi}_{\ell-1} 1_{V_i^\ell}$.

(ii) Each agent i has taken new measurements $\phi(p_i^\ell)$ and $\phi(C_{V_i^\ell})$, after having moved from p_i^ℓ to $C_{V_i^\ell}$.

Under these assumptions, agent i will be able to compute:

$$N_i^\ell(0) = (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell)) \left(\phi(C_{V_i^\ell}) M_{V_i^\ell} - \int_{V_i^\ell} \overline{\phi}_{\ell-1} dq \right) + C\,,$$

$$D_i^\ell(0) = (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell))^2 M_{V_i^\ell} + C\,,$$

where $C = \sigma^2 M_Q \Pi_{s=1}^{\ell-1}(1 - W_s^*)^2$. After K_ℓ communication rounds take place, agents are able to update the values of $N_i^\ell(m)$ and $D_i^\ell(m)$, $m \in \{1,\dots,K_\ell\}$, obtain the positions of new Voronoi neighbors at time $t_{\ell+1}$, and new information that allows them to compute $\overline{\phi}_\ell 1_{V_i^{\ell+1}}$.

The update of $N_i^\ell(m)$, $D_i^\ell(m)$, is given in Theorem 5. The computation of $\overline{\phi}_\ell 1_{V_i^{\ell+1}}$ is explained in Theorem 6.

Theorem 5: Let $\ell \in \mathbb{N}$ be fixed. Consider the particular case of a time schedule $\overline{\mathbb{T}} = \{t_{rm}\}$ with $r \in \{1,\dots,\ell\}$ and infinite number of communication rounds after t_ℓ, $\{s_\ell^m\}_{m\in\mathbb{N}}$. At time t_ℓ suppose that Assumptions (i) and (ii) hold for each agent. Denote by $\mathcal{G}(m)$, $m \in \mathbb{N}$ the communication graphs at times $t_{\ell m}$. Define the consensus algorithms:

$$N_j^\ell(m+1) = \sum_{i=1}^n F_j^i(m) N_i(m),$$

$$D_j^\ell(m+1) = \sum_{i=1}^n F_j^i(m) D_i(m),$$

Here $F(s)$ is a stochastic matrix such that $F(m) = (I + \mathrm{Deg}(m))^{-1}(I + A(m))$, where $A(m)$ (resp. $\mathrm{Deg}(m)$) is the adjacency matrix (resp. the degree matrix) of the graph $\mathcal{G}(m)$. If there exists a $M > 0$ such that for any $m_0 \in \mathbb{N}$ the union $\cup_{m=m_0}^{m_0+M} \mathcal{G}(m)$ is connected, then $N_j^\ell(m)/D_j^\ell(m) \to W_1^*$ exponentially fast as $m \to +\infty$, for all $j \in \{1,\dots,n\}$.

Proof: The proof is a consequence of the convergence properties of consensus algorithms, see [2], [14]. For undirected graphs, the consensus limit values are given by $\frac{1}{n}\sum_{i=1}^n N_i^\ell(0)$ and $\frac{1}{n}\sum_{i=1}^n D_i^\ell(0)$ respectively and then $N_j^\ell(m)/D_j^\ell(m)$ converges to $(\sum_{i=1}^n N_i^\ell(0))/(\sum_{i=1}^n D_i^\ell(0)) = W_\ell^*$.

The exponential convergence nature of consensus algorithms is also a well known fact, see [14], and it depends on the degree of connectivity of the graphs $\mathcal{G}(m)$ and the number of agents n. In fact, a weaker notion of *strongly rooted graph* and weaker assumptions on connectivity allow to deduce an exponential convergence rate to some consensus value, see [14]. ■

The best possible decentralization procedure would allow each vehicle to know exactly $\overline{\phi}_\ell$ over Q, for all $\ell \in \mathbb{N}$. A less stringent procedure is one where vehicles just have information of $\overline{\phi}_\ell$ on their Voronoi regions $V_i^{\ell+1}$, $\ell \in \mathbb{N}$. In fact, this is at least necessary for each vehicle to compute N_i^ℓ and D_i^ℓ. The degree of decentralization is further determined via the following class of proximity graphs.

We define the graph $\mathcal{G}^*(t_\ell) = (\{1,\dots,n\}, \mathcal{E}^*(t_\ell))$, via the set of *neighbors* of $i \in \{1,\dots,n\}$ at time $t_\ell \in \overline{\mathbb{T}}$:

$$\mathcal{N}_i^*(\ell) = \{j \in \{1,\dots,n\} \mid V_j^{\ell-1} \cap V_i^\ell \neq \emptyset\}, \quad \ell \in \mathbb{N}.$$

Observe that, contrary to the Delaunay graph, these are only *directed* graphs. In order to compute $\overline{\phi}_\ell 1_{V_i^{\ell+1}}$ each agent needs to communicate with neighbors $\mathcal{N}_i^*(\ell+1)$ and the Delaunay neighbors $\mathcal{N}_i(\ell+1)$. This is discussed in the following theorem.

Theorem 6: Let $\{p_1, \dots, p_n\}$ be a robotic network moving over a region Q. Suppose they can synchronously take new measurements of a field $\phi : Q \longrightarrow \mathbb{R}$ as specified by a time scheduling sequence $\overline{\mathbb{T}}$. Assume also that agents can compute the gains W_ℓ^*, $t_\ell \in \overline{\mathbb{T}}$, in a distributed manner, e.g., as in Theorem 5. Then, the coordinated computation of $\overline{\phi}_\ell$ can be done as a sum of contributions $\overline{\phi}_\ell(q) = \sum_{i=1}^n \overline{\phi}_\ell^i(q)$, for all $q \in Q$, as long as each agent can communicate with neighbors in the graph $\mathcal{G}^*(t_\ell)$, $t_\ell \in \overline{\mathbb{T}}$. Here $\overline{\phi}_\ell^i(q) = f_\ell^i(q) \cdot 1_{V_i^\ell}(q)$ is maintained by each vehicle, where:

$$f_\ell^i(q) = \sum_{j \in \mathcal{N}_i^*(\ell)} f_{\ell-1}^j(q) 1_{V_j^{\ell-1}}(q) + W_\ell^*(\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_j^\ell)),$$

$\overline{\phi}_{\ell-1}(p_j^\ell) = f_{\ell-1}^{k_j}(p_j^\ell)$ with $p_j^\ell \in V_{k_j}^{\ell-1}$, for all $\ell \in \mathbb{N}$, and $f_0^j(q) = \phi(p_j^0)$, $j \in \{1, \dots, n\}$.

Proof: Let $\ell = 1$. By definition, see Eq. 1:

$$\overline{\phi}_1 = \overline{\phi}_0 + W_1^*(\overline{Q}_1\phi - \overline{Q}_1\overline{\phi}_0)$$

Since $\{V_j^\ell\}_{j=1}^n$ is a partition of Q for all $\ell \in \mathbb{N}$, then:

$$\overline{\phi}_0(q) = \sum_{k=1}^n \phi(p_k^0) 1_{V_k^0}(q) = \sum_{j=1}^n \Big(\sum_{k=1}^n \phi(p_k^0) 1_{V_k^0}(q)\Big) 1_{V_j^1}(q).$$

Since $1_{V_k^0}(q) 1_{V_j^1}(q) \neq 0$ iff $k \in \mathcal{N}_j^*(1)$, we have:

$$\overline{\phi}_0(q) = \sum_{j=1}^n \Big(\sum_{k \in \mathcal{N}_j^*(1)} \phi(p_k^0) 1_{V_k^0}(q)\Big) 1_{V_j^1}(q).$$

Using this fact, and Eq. 1, we obtain:

$$\overline{\phi}_1(q) = \sum_{j=1}^n f_1^j(q) 1_{V_j^1}(q)$$

with $f_1^j(q) = \sum_{k \in \mathcal{N}_j^*(1)} \phi(p_k^0) 1_{V_k^0}(q) + W_1^*\big(\phi(p_j^1) - \overline{\phi}_0(p_j^1)\big)$. It is not difficult to see that the general case follows easily by induction. ■

In other words, to maintain a data-base representation of ϕ in its current region each vehicle needs to communicate with others that were covering this portion of the space before.

Knowledge of the regions $V_i^{\ell-1}$, $i \in \mathcal{N}_j^*(\ell)$ can be obtained by knowing the positions of the Voronoi neighbors of vehicle i at time $\ell - 1$. The cardinal of the set $\mathcal{N}_j^*(\ell)$ will depend on the given motion coordination algorithm. For example, if the motion of vehicles is prescribed to the centroids $C_{V_i^{\ell-1}}$, it is reasonable to expect that $\mathcal{N}_j^*(\ell) \cap \mathcal{N}_j(\ell - 1) \neq \emptyset$. In general we can not guarantee the graphs are connected. In order to apply Theorem 5 one should guarantee communication over a larger undirected graphs containing $\mathcal{G}^*(\ell)$, $\ell \in \mathbb{N}$.

V. Conclusions

We have presented a simple refinement interpolation scheme for static field representation and conditions for its possible decentralization. Future work will address the problem of how to limit the resolution of the filter (considering a finite number of subregions for each V_i^ℓ) and the combination of the approach with specific motion coordination plans.

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