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Practical rendezvous through modified circumcenter algorithms

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***Abstract*— We present modified nonlinear circumcenter algorithms to achieve "practical" rendezvous when agents take noisy measurements of their neighbors' positions. Assuming a uniform probability distribution of the noise in a disk about the true position, we analyze the algorithms in 1D. In particular, we provide a characterization of the practical stability ball via deterministic and stochastic analysis tools. The higher dimensional cases are discussed in simulation and we propose modified "parallel" circumcenter algorithms that can be used with guaranteed performance.**

I. Introduction

Distributed cooperative systems are attracting an intense research activity in the last years, see for instance [1]. As a consequence of this, a wealth of algorithms is being proposed together with novel analysis tools to evaluate their performance. When doing this, one important aspect to consider is that of robustness. Ideally, a characterization of what are the typical degraded behaviors under the algorithm should be provided, together with some discussion how those are affected by the network size. If necessary, the algorithm should be modified to guarantee different robustness aspects.

Motivated by this, we discuss how the nonlinear Circumcenter Algorithm, see [2], can be made robust with respect to measurement noise. This complements the simulation analysis performed in [2] that showed good performance of the algorithm under noise, and the work [3] which considered an asynchronous version of the algorithm. As we discuss later, a deterministic approach is not enough to explain the behavior of the algorithm under different communication graphs.

The Circumcenter Algorithm was first proposed by Suzuki *et al* in [2] as a memoryless algorithm that allows a network of robotic agents rendezvous to a common location of the space. The algorithm was further studied in [4], [5], and asynchronous versions of the algorithm are presented in [6], [3].

With regards to the type of results presented here, the papers [7], [8], [9] do a similar study of how consensus algorithms are robust to measurement, communication noise, and quantization errors. However, the type of algorithms considered here are based on the nonlinear Circumcenter Algorithm, while those papers consider linear consensus algorithms.

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To be more precise, the contributions of this paper can be summarized as follows. Assuming that each agent measures neighbors' positions up to an error σ, (according to a uniform distribution in a disk of radius σ), we propose two possible modifications of the standard Circumenter Algorithm. The first version restricts the constraint set to guarantee connectivity of the network. The second version assumes that agents filter their measurements of neighbors to make sure that they are still within the connectivity radius r.

We analyze both algorithms in a 1D space. The implementation of the algorithm over the r-disk graph allows to derive a deterministic analysis that shows convergence to a ball of diamater twice the error σ independently of the number of agents in the network. This is a type of ISS result that requires that $r > 7\sigma$ in the first version of the algorithm. As shown in simulations, the analysis can not be carried over for graphs different from the r-disk graph. A stochastic analysis is necessary to extend the determistic result to any graph, for $r > 3\sigma$, but at the cost of having a convergence only in probability one.

The proofs presented here make a fundamental use of the definition of the circumcenter in 1D, much simpler than in higher dimensional spaces. However, simulations seem to indicate that the algorithms work also in higher dimensions. Under the assumption that all agents in the network have knowledge of a common orientation framework, we extend the new algorithms as modified Parallel Circumcenter Algorithms. These algorithms can be used with guaranteed performance.

The paper is organized as follows. Section II introduces premilinary notions, the standard Circumcenter Algorithm and Parallel Circumcenter Algorithm. Section III introduces two possible modifications of the Circumcenter Algorithm to cope with noise. Finally, the last section presents some simulations of the algorithms.

II. Preliminaries and algorithm description

We review here some notation for standard geometric objects; for additional information we refer the reader to [10] and references therein. We then recall the Circumcenter and Parallel Circumcenter Algorithms as discussed in [2], [4], [11]. In the last part of this section, we present the new Modified Circumcenter Algorithms.

A. Basic geometric notions and notation

For a bounded set $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, we let $\mathrm{co}(S)$ denote the convex hull of S. For $p, q \in \mathbb{R}^d$, we let $(p, q) = \{\lambda p + (1-\lambda)q \mid \lambda \in (0,1)\}$ and $[p, q] = \mathrm{co}(\{p, q\})$ denote the *open* and *closed segment* with extreme points p and q, respectively. For a bounded set $S \subset \mathbb{R}^d$, we let $\mathrm{CC}(S)$ and $\mathrm{CR}(S)$ denote the *circumcenter* and *circumradius* of S, respectively, that is, the center and radius of the smallest-radius d-sphere enclosing S. The computation of the circumcenter and circumradius of a bounded set is a strictly convex problem and in particular a quadratically constrained linear program. For $p \in \mathbb{R}^d$, $B(p, r)$ and $D(p, r)$ denote the *open* and *closed ball* of center p and radius $r \in \mathbb{R}_{>0}$, respectively. Here, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ will denote the positive and the nonnegative real numbers, respectively.

Let $\mathbb{F}(\mathbb{R}^d)$ be the collection of finite point sets in $\mathbb{R}^d$; we shall denote an element of $\mathbb{F}(\mathbb{R}^d)$ by $\mathcal{P} = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$, where $p_1, \dots, p_n$ are distinct points in $\mathbb{R}^d$. Let $\mathbb{G}(\mathbb{R}^d)$ be the set of undirected graphs whose vertex set is an element of $\mathbb{F}(\mathbb{R}^d)$. A *proximity graph function* $\mathcal{G} \colon \mathbb{F}(\mathbb{R}^d) \to \mathbb{G}(\mathbb{R}^d)$ associates to a point set $\mathcal{P}$ an undirected graph with vertex set $\mathcal{P}$ and edge set $\mathcal{E}_{\mathcal{G}}(\mathcal{P})$, where $\mathcal{E}_{\mathcal{G}} \colon \mathbb{F}(\mathbb{R}^d) \to \mathbb{F}(\mathbb{R}^d \times \mathbb{R}^d)$ has the property that $\mathcal{E}_{\mathcal{G}}(\mathcal{P}) \subseteq \mathcal{P} \times \mathcal{P} \setminus \mathrm{diag}(\mathcal{P} \times \mathcal{P})$ for any $\mathcal{P}$. Here, $\mathrm{diag}(\mathcal{P} \times \mathcal{P}) = \{(p, p) \in \mathcal{P} \times \mathcal{P} \mid p \in \mathcal{P}\}$. In other words, the edge set of a proximity graph depends on the location of its vertices. General properties of proximity graphs, basics on graph theory and examples can be found in [10], [12], [13].

In particular, we will make use of the r-disk proximity graph $\mathcal{G}_{\text{disk}}(r)$, for $r \in \mathbb{R}_{>0}$ and over a set of vertices $\mathcal{P}$. In this graph, two agents $p_i, p_j \in \mathcal{P}$ are neighbors iff $\|p_i - p_j\| \leq r$. We denote the set of neighbors of agent $p_i \in \mathcal{P}$ in a proximity graph by:

$$\mathcal{N}_i(\mathcal{G}) = \{j \in \{1, \dots, n\} \mid (p_i, p_j) \in \mathcal{E}_{\mathcal{G}}(\mathcal{P})\}.$$

For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda(q_0, q_1, Q)$ denote the solution of the strictly convex problem:

$$\begin{aligned} &\text{maximize} \ \ \lambda \\ &\text{subject to} \ \ \lambda \leq 1, \ (1-\lambda)q_0 + \lambda q_1 \in Q. \end{aligned} \tag{1}$$

Note that this convex optimization problem has the following interpretation: move along the segment from q_0 to q_1 the maximum possible distance while remaining in Q. Under the stated assumptions the solution exists and is unique.

B. Circumcenter Algorithms

The following is an informal description of the Circumcenter Algorithm defined for a proximity graph $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$, with $r \in \mathbb{R}_{>0}$.

Standard Circumcenter Algorithm:

> Each agent performs: (i) it detects its neighbors according to $\mathcal{G}$; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself, and (iii) it moves toward this circumcenter while maintaining connectivity with its neighbors.

This algorithm was originally introduced in [2] and its asymptotic convergence is guaranteed as proven in [5] for any switching sequence of proximity graphs that contains a strongly connected graph every l instants of time, for some fixed $l \in \mathbb{N}$. The asynchronous behavior of the algorithm for was analyzed in [6], [3].

In [11], it was proven that, when implemented over a 1D space, it is not necessary to enforce the connectivity constraint. In other words, step (iii) can be rephased as "(iii) agent moves to the circumcenter of neighbors". Assuming that agents have knowledge of a frame with a common orientation, we can extend the 1D algorithm to arbitrary dimensions by means of a Circumcenter Algorithm implemented in parallel as follows.

Parallel Circumcenter Algorithm:

> Each agent performs: (i) it detects its neighbors according to $\mathcal{G}$; (ii) it projects the detected positions to each axis of its frame; (iii) it computes the circumcenters of each of the projected sets of positions on each axis (iii) it moves to the point whose coordinates are given by each of those circumcenters.

For formal descriptions of these algorithms written in pseudocode we refer the reader to [5], [11].

C. Modified Circumcenter Algorithm

Assume now that each agent i is able to detect a perturbed position, $\overline{p}_j^i \in D(p_j, \sigma)$, of agent j, only. In other words, p_j is the true position of agent j that agent i measures as $\overline{p}^i_j$ up to an error $0 < \sigma < r$. With a slight abuse of notation we will denote $\overline{p}^i_i = p_i$. In what follows, we assume a centered detection probability over the disk $D(p_j, \sigma)$; that is, $E[\overline{p}^i_j] = p_j$, $\forall\, j \in \mathcal{N}_i(\mathcal{G})$. In particular, this is satisfied by the uniform probability distribution over $D(p_j, \sigma)$ that we consider here. Given a set of agents $\mathcal{P} \subseteq \mathbb{R}$, we will denote by $\overline{p}^i_m = \min\{\overline{p}^i_j \mid j \in \mathcal{N}_i(\mathcal{G}) \cup \{i\}\}$ (resp. $p^i_m = \min\{p_j \mid j \in \mathcal{N}_i(\mathcal{G}) \cup \{i\}\}$) and $\overline{p}^i_M = \max\{\overline{p}^i_j \mid j \in \mathcal{N}_i(\mathcal{G}) \cup \{i\}\}$ (resp. $p^i_M = \max\{p_j \mid p_j \in \mathcal{N}_i(\mathcal{G}) \cup \{i\}\}$).

To still guarantee connectivity, a possibility is to change the standard Circumcenter Algorithm by restricting the constraint set where agents are allowed to move.

Name:	Modified Circ. Algorithm (MCA) v. 1
Goal:	All agents practically rendezvous
Assumes:	(i) $r \in \mathbb{R}_+$ is sensing radius
	(ii) $\sigma < r$ is the sensing error
	(iii) Agents are initially connected by $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$

For $i \in \{1, \dots, n\}$, agent i executes at each time instant in $\mathbb{N}$:

1: acquire $\{\overline{p}^i_{i_1}, \dots, \overline{p}^i_{i_k}\}$, s.t. $\overline{p}^i_j \in D(p_j, \sigma)$ is within error σ of true position p_j, for each $i_\ell \in \mathcal{N}_i(\mathcal{G})$
2: compute $\overline{\mathcal{M}}_i := \{\overline{p}^i_{i_1}, \dots, \overline{p}^i_{i_k}\} \cup \{p_i\}$, $i_\ell \in \mathcal{N}_i(\mathcal{G})$
3: compute $\overline{Q}_i := \cap_{q \in \overline{\mathcal{M}}_i} D\Big(\dfrac{q + p_i}{2}, \dfrac{r - \sigma}{2}\Big) \cup \{p_i\}$
4: compute $\lambda^*_i := \lambda(p_i, \mathrm{CC}(\overline{\mathcal{M}}_i), \overline{Q}_i)$
5: set $u_i := \lambda^*_i(\mathrm{CC}(\overline{\mathcal{M}}_i) - p_i)$, i.e., move from p_i to $(1 - \lambda^*_i)p_i + \lambda^*_i\, \mathrm{CC}(\overline{\mathcal{M}}_i)$

Here, we are implicitly assuming that each agent has knowledge of the committed error σ. An alternative to this algorithm is MCA v. 2 that filters the values $\overline{p}^i_j$. That is, if $\|p_j - p_i\| \le r$ then $\overline{p}^i_j$ is taken so that $\|\overline{p}^i_j - p_i\| \le r$. In this case $\overline{Q}_i$ can be defined in the standard way as:

$$\overline{Q}_i = \cap_{q \in \overline{\mathcal{M}}_i} D\Big(\frac{q + p_i}{2}, \frac{r}{2}\Big)\,.$$

Given $\mathcal{P} = \{p_1, \dots, p_n\}$, from now on we will use the notation p^+_i for the next position of p_i, $i \in \{1, \dots, n\}$ under any MCA.

Observe that while the intersection of disks

$$\overline{D}_i = \bigcap_{q \in \overline{\mathcal{M}}_i} D\Big(\frac{q + p_i}{2}, \frac{r - \sigma}{2}\Big)$$

in MCA v. 1 might be empty, the set $\overline{Q}_i$ is guaranteed to be nonempty with the inclusion of $\{p_i\}$. Note that it could also happen that $p_i \notin \overline{D}_i$. The set $\overline{Q}_i$ is defined to guarantee connectivity as stated in the next lemma.

Lemma 1 (Connectivity Maintenance): Consider $\mathcal{G}_{\text{disk}}(r)$ for some $r > 0$ and let $(p_i, p_j) \in \mathcal{E}_{\mathcal{G}_{\text{disk}}(r)}(\mathcal{P})$. Then, under the MCA, $\|p^+_i - p^+_j\| \le r$.

Proof: This fact can be easily verified. ■

In 1D, the MCA v. 1 satisfies the following properties.

Lemma 2: (i) $p_i \in \overline{D}_i$ if and only if $\|p_i - \overline{p}^i_M\| \le r - \sigma$ and $\|p_i - \overline{p}^i_m\| \le r - \sigma$.
(ii) $\overline{D}_i \neq \emptyset$ if and only if $\|\overline{p}^i_M - \overline{p}^i_m\| \le 2(r - \sigma)$.

Lemma 3: Assume that $q_1 < \mathrm{CC}(\overline{\mathcal{M}}_i) < q_2$ for agent $i \in \{1, \dots, n\}$ and some $q_1, q_2 \in \mathbb{R}$. Then, under the MCA v. 1,

(i) If $\overline{D}_i \neq \emptyset$ and

$$\frac{\overline{p}^i_m + p_i}{2} + \frac{r - \sigma}{2} - q_1 > 0\,, \tag{2}$$

holds, then $p^+_i > q_1$.

(ii) If $\overline{D}_i \neq \emptyset$ and

$$\frac{\overline{p}^i_M + p_i}{2} - \frac{r - \sigma}{2} - q_2 < 0\,. \tag{3}$$

then $p^+_i < q_2$.

(iii) If $\overline{D}_i = \emptyset$ and $q_1 < p_i < q_2$, then $q_1 < p^+_i < q_2$.

Proof: For reasons of space we omit the proof of this fact. ■

III. Deterministic Analysis of the Modified Circumcenter Algorithm in 1D

In this section we include a deterministic analysis of the MCA v. 1 in 1D. The MCA v. 2 analysis becomes a particular case as it uses less restrictive constraint sets.

Theorem 4: Let $p_1(0), \dots, p_n(0)$ be the initial positions of a robotic network in $\mathbb{R}$. Suppose the agents are initially connected by $\mathcal{G}_{\text{disk}}(r)$ for a sensing radius $r > 0$. Let $\sigma > 0$ be the sensing error radius and $\{P_m = (p_1(m), \dots, p_n(m))\}_{m \in \mathbb{N} \cup \{0\}}$ a sequence of positions obtained by applying the MCA v. 1 with $\mathcal{G}_{\text{disk}}(r)$. Then, if $r > 7\sigma$, we have $P_m \to \mathcal{S}_D$, as $m \to \infty$, where

$$\mathcal{S}_D = \{P \in \mathbb{R}^n \mid \mathrm{diam}(P) \le 2\sigma\}\,.$$

Proof: For space reasons we omit the proof of this fact. We refer the reader to a forthcoming extended version of this paper in `http://flyingv.ucsd.edu/sonia`

■

Remark 5: This result holds independently of the number of agents in the network, which in particular does not affect the size of the practical stability ball. As we show in simulations later, the ball does wander in space by the effect of noise. The theorem gives only sufficient conditions for decreasing the diameter strictly after two time steps. Simulations also show convergence for smaller ratios r/σ.

IV. Stochastic analysis of the Modified Circumcenter Algorithm in 1D

In this section we present a stochastic analysis of the MCA in 1D. This allows to trade in a weaker convergence condition in probability one by a less stronger assumption on r/σ and the possibility of using any graph $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$. The analysis also becomes much simpler by using the following supermartingale convergence theorem (Doob's theorem) taken from [14].

Theorem 6 (Supermartingale Convergence Theorem): Suppose that X_t is a nonnegative random variable such that $E[X_1] < +\infty$. Let $\mathcal{F}_t$ denote the history of process X_t up to time t. If

$$E[X_{t+1} \big| \mathcal{F}_t] \leq X_t, \qquad \text{w.p.1}$$

then X_t converges to a limit w.p.1.

Using this theorem we can obtain the main result in this section. Before stating it, we include a useful lemma.

Lemma 7: Consider a set of random variables $\{A_s\}_{s\in\{1,\dots,m\}}$ taking real values. Then,

(i) $E[\max_s A_s] = \max_s E[A_s]$
(ii) $E[\min_s A_s] = \min_s E[A_s]$
(iii) If $A_1 \leq A_2$ then $E[A_1] \leq E[A_2]$.

Proof: We can derive (i) by observing that

$$E[A_s] \leq E[\max_s A_s] = E[A_*] \leq \max E[A_s],$$

and similarly (ii) can be obtained. Finally, (iii) is a consequence of the definition of the expectation. ■

Theorem 8: Let $p_1(0), \dots, p_n(0)$ be the initial positions of a robotic network in $\mathbb{R}$. Suppose the agents are initially connected by $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$ for a sensing radius $r > 0$. Let $\sigma > 0$ be the sensing error radius and let $\{P_m = (p_1(m), \dots, p_n(m))\}_{m\in\mathbb{N}\cup\{0\}}$ denote a sequence of positions obtained by applying MCA with $\mathcal{G}$. Then, if $r > 3\sigma$, we have $E[P_m] \to 0$ as $m \to \infty$.

Proof: For space reasons we omit the proof of this fact. We refer the reader to a forthcoming extended version that can be downloaded at `http://flyingv.ucsd.edu/sonia` ■

Remark 9: The proof just presented makes use of the standard Circumcenter Algorithm proof taken from [5] and relies on the simpler definition of circumenter in 1D. As in the latter paper, this proof is valid for a fixed graph $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$ and can also be extended to allow certain switching of graphs $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$. We refer the reader to a forthcoming report for the detailes on this fact. The previous proof uses in a fundamental way that $E[\overline{\mathrm{CC}}_i|t] = \mathrm{CC}_i$. We conjecture that in higher dimensions, at least we have that $E[\overline{R}_i|t] = R_i$, where $\overline{R}_i$ are the perceived local circumradius by agent i and R_i is the actual local circumradius.

Proposition 10: The set of limit configurations described in the previous theorem can be characterized to be contained in:

(i) $D\Big(\frac{1}{2}(\max p_i + \min p_i), \frac{1}{2}(r-\sigma)\Big)$ for $\mathcal{G}$.
(ii) $D\Big(\frac{1}{2}(\max p_i + \min p_i), \sigma\Big)$ for $\mathcal{G}_{\text{disk}}(r)$.

Proof: If $E[\text{diam}(p_1^+, \dots, p_n^+)|(p_1, \dots, p_n)] = 0$,

$$\begin{aligned} 0 =& E[\text{diam}(p_1^+, \dots, p_n^+)|\mathcal{P}] = \max_{i,j} E[\|p_i^+ - p_j^+\||\mathcal{P}] \\ &\geq \max_{i,j} \|E[p_i^+|\mathcal{P}] - E[p_j^+|\mathcal{P}]\| \,. \end{aligned}$$

That is, $E[p_i^+|\mathcal{P}] = E[p_j^+|\mathcal{P}] = p$ for all $i, j \in \{1, \dots, n\}$. In particular this implies that

$$p \in \bigcap_{j\in\{1,\dots,n\}} (\{p_j\} \cup D_j) = \bigcap_{j\in\{1,\dots,n\}} \{p_j\} \cup \bigcap_{j\in\{1,\dots,n\}} D_j \tag{4}$$

and that the above intersection is nonempty. Since

$$\bigcap_{j\in\{1,\dots,n\}} D_j = \left[p_n - \frac{r-\sigma}{2}, p_1 + \frac{r-\sigma}{2}\right]$$

intersection (4) is nonempty if and only if:

$$p_i = p_j \quad \forall i, j \qquad \text{or} \quad p_n - p_1 \leq r - \sigma \,. \tag{5}$$

Both conditions imply that we have reached a ball of diameter $r - \sigma$. In fact, if CC denotes the circumcenter of the set of all agents, we have that:

$$p_i \in D\Big(\mathrm{CC}, \frac{r-\sigma}{2}\Big) \subseteq \left[p_n - \frac{r-\sigma}{2}, p_1 + \frac{r-\sigma}{2}\right], \tag{6}$$

$\forall\, i \in \{1, \dots, n\}$. Observe this is valid for any graph $\mathcal{G}$.

Now consider the particular case of $\mathcal{G}_{\text{disk}}(r)$. Condition (5) implies that p_1 and p_n are connected and we have reached the complete graph. From the set content (6) we also see that it will not be necessary to enforce the constraint in the Modified Circumcenter Algorithm v. 1 since it automatically holds. Therefore we have that $p_i^+ = \overline{\mathrm{CC}}_i$ and $p = E[\overline{\mathrm{CC}}_i|\mathcal{P}] = \mathrm{CC}$, $\forall\, i \in \{1, \dots, n\}$. Since $\overline{\mathrm{CC}}_i \in D\Big(\mathrm{CC}_i, \sigma\Big)$, then it must be that $p_i^+ \in D\Big(\mathrm{CC}(p_1, \dots, p_n), \sigma\Big)$, $\forall\, i \in \{1, \dots, n\}$. ■

V. Simulations

As with the standard Circumcenter Algorithm, the 1D Modified Circumcenter Algorithm can be extended in "parallel" to any dimension by means of a modification of the parallel circumcenter algorithm. The following is a formal description of this procedure. It assumes knowledge of a frame $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$,with $\mathbf{e}_i \in \mathbb{R}^d$, for all $i \in \{1, \dots, n\}$. We denote $\pi_a : \mathbb{R}^d \to \mathbb{R}$ the projection $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_d\mathbf{e}_1 \to x_a$ relative to $\mathbf{e}_a$ for $a \in \{1, \dots, d\}$.

Name: Modified Parallel C. A. (MPCA) v. 1
Goal: All agents practically rendezvous
Assumes: (i) $r \in \mathbb{R}_+$ is sensing radius
(ii) $\sigma < r$ is the sensing error
(iii) Agents are initially connected by $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$
(iv) Knowledge about a common reference frame $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_d\} \subseteq \mathbb{R}^{d \times d}$

For $i \in \{1, \dots, n\}$, agent i executes at each time instant in $\mathbb{N}$:

1: acquire $\{\overline{p}^i_{i_1}, \dots, \overline{p}^i_{i_k}\}$, s.t. $\overline{p}^i_j \in D(p_j, \sigma)$ is within error σ of true position p_j, for each $i_\ell \in \mathcal{N}_i(\mathcal{G})$
2: compute $\overline{\mathcal{M}}^a_i := \{\pi_a(\overline{p}^i_{i_1}), \dots, \pi_a(\overline{p}^i_{i_k})\} \cup \{\pi_a(p_i)\}$, $i_\ell \in \mathcal{N}_i(\mathcal{G})$, $a \in \{1, \dots, d\}$.
3: compute $\overline{Q}^a_i := \cap_{q \in \overline{\mathcal{M}}^a_i} D\Big(\dfrac{q + \pi_a(p_i)}{2}, \dfrac{r - \sigma}{2}\Big) \cup \{\pi_a(p_i)\}$
4: compute $\lambda^{a,*}_i := \lambda(\pi_a(p_i), \mathrm{CC}(\overline{\mathcal{M}}^a_i), \overline{Q}^a_i)$
5: set $u^a_i := \lambda^{a,*}_i(\mathrm{CC}(\overline{\mathcal{M}}^a_i) - \pi_a(p_i))$, i.e., move from p_i to a point with components $(1 - \lambda^{a,*}_i)\pi_a(p_i) + \lambda^{a,*}_i \,\mathrm{CC}(\overline{\mathcal{M}}^a_i)$, for $a \in \{1, \dots, d\}$.

The convergence of this algorithm is guaranteed by straightforward extensions of Theorems 4 and 8.

A. Simulations

Figure 1 shows a run of the MCA v. 2 for $\mathcal{G}_{\text{disk}}(r)$ in 2D, 15 agents and 300 time steps. Here $r = 6$ and $\sigma = 3$. The connectivity of the group of the 15 agents is shown in the left box of Figure 1 while its evolution is shown in the right box. As it can be seen in this figure and the diameter evolution in Figure 3, the algorithm behaves even better than expected from the 1D analysis. There is a slight wandering of the practical stability ball. This behavior is representative of what we have seen in many repeated simulations with different initial conditions and relations r/σ.

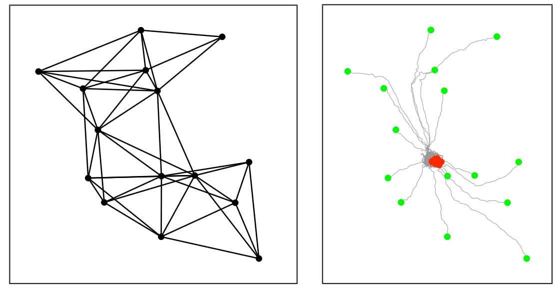

Fig. 1. MCA v. 2 for $\mathcal{G}_{\text{disk}}(r)$, $r = 3$, $\sigma = 1$ in 2D

Although we do not include this result here, we have been able to check that in 2D the circumcenter $\overline{\mathrm{CC}}$ of perturbed positions $\bar{p}_i \in D(p_i, \sigma)$ satisfies a bound $\|\overline{\mathrm{CC}} - \mathrm{CC}\| = O(\sqrt{R\sigma})$, where R is the circumradius of the unperturbed positions p_i, $i \in \{1, \dots, n\}$. In fact a bound $\|\overline{\mathrm{CC}} - \mathrm{CC}\| = \sqrt{\frac{R\sigma}{2}}$ can indeed be reached by some specific configuration and when $R > 2\sigma$. We will summarize this results in a forthcoming publication.

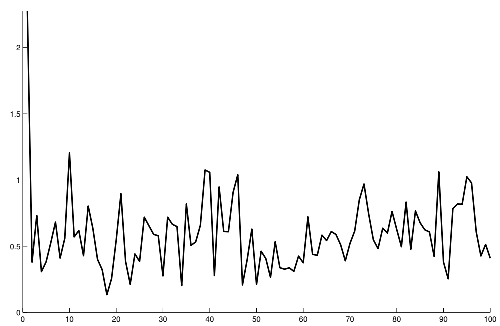

Fig. 2. Diameter evolution of a network of 4 agents under the MCA v. 1 for $\mathcal{G}_{\text{disk}}(r)$, $r = 3$, $\sigma = 1$ in 1D.

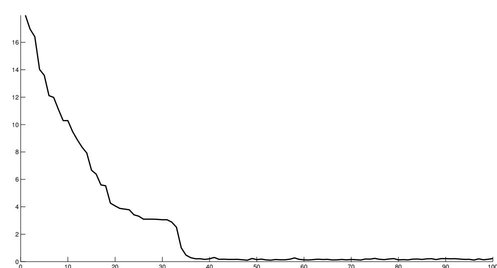

Fig. 3. Diameter evolution of a network of 50 agents under the MCA v. 1 for $\mathcal{G}_{\text{disk}}(r)$, $r = 3$, $\sigma = 1$ in 1D.

Simulations of the evolution of the diameter under the MCA v. 1 for $\mathcal{G}_{\text{disk}}(r)$ are shown in Figures 2 and 3 for a network of 4 and 50 agents. In general we observe that

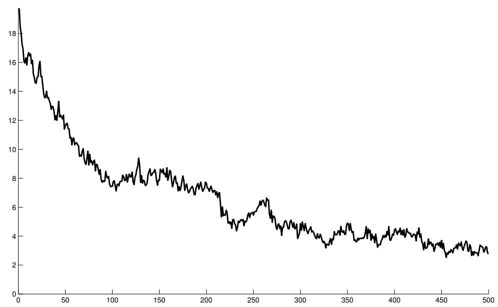

Fig. 4. Diameter evolution of 30 agent network for MCA v. 1 for a fixed graph $\mathcal{G}$ which corresponds to the Limited Delaunay graph of the initial positions, $r = 3$ and $\sigma = 1$ in 1D.

the smaller the group of agents, the increased wandering of the stability ball and the larger its diameter. When the number of agents is increased a filtering effect is produced which favors the final outcome. Note that, for the case of 50 agents, the diameter function remains constant for some period of time. This is due to the constraint enforcement that does not allow agents to move, which can happen when $r - \sigma$ is small. Since the information about the neighbors positions changes randomly in time according to a uniform distribution, it is clear that eventually the constraint sets will become nonempty and agents will be able to move. This is why we believe the MCA works in fact for any $r > \sigma$. A simulation of MCA v. 1 for a fixed graph $\mathcal{G} \subseteq \mathcal{G}_{\mathrm{disk}}(r)$ and 30 agents is shown in Figure 4. Convergence here is much slower due to the fact that each agent has only two neighbors. Note also that the diameter may increase at any time, so a deterministic analysis like the for $\mathcal{G}_{\mathrm{disk}}(r)$ is no longer feasible.

VI. CONCLUSIONS

We have introduced two possible modifications of the Circumcenter Algorithm to cope with noisy measurements of neighbors positions. We are currently working on the extension of the results to 2D dimensions and the possibility of using a family of switching graphs.

VII. ACKNOWLEDGMENTS

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REFERENCES

[1] V. Kummar, N. Leonard, and A. Morse, eds., *Proc. of the 2003 Block Island Workshop on Cooperative Control*, vol. 309 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 2004.

[2] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Transactions on Robotics and Automation*, vol. 15, no. 5, pp. 818–828, 1999.

[3] J. Lin, A. Morse, and B. Anderson, "The Multi-Agent Rendezvous Problem –Part 2: The Asynchronous Case," *SIAM J. Control and Optimization*, 2005. Submitted.

[4] J. Lin, A. Morse, and B. Anderson, "The Multi-Agent Rendezvous Problem –Part 1: The Synchronous Case," *SIAM J. Control and Optimization*, 2005. To appear.

[5] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1289–1298, 2006.

[6] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of asynchronous oblivious robots with limited visibility," in *STACS 2001, 18th Annual Symposium on Theoretical Aspects of Computer Science (Dresden, Germany)* (A. Ferreira and H. Reichel, eds.), vol. 2010 of *Lecture Notes in Computer Science*, pp. 247–258, New York: Springer Verlag, 2001.

[7] L. Schenato and S. Zampieri, "Optimal rendezvous control for randomized communication topologies," in *IEEE Conf. on Decision and Control*, (San Diego), pp. 4339–4344, December 2006.

[8] L. Xiao, S. Boyd, and S. J. Kim, "Distributed average consensus with least-mean-square deviation," *Journal of Parallel and Distributed Computing*, vol. 67, no. 1, pp. 33–46, 2007.

[9] D. B. Kingston, W. Ren, and R. W. Beard, "Consensus algorithms are input-to-state stable," in *American Control Conference*, (Portland, OR), pp. 1686–1690, June 2005.

[10] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, *Computational Geometry: Algorithms and Applications*. New York: Springer Verlag, 2 ed., 2000.

[11] S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli, "On synchronous robotic networks – Part II: Time complexity of rendezvous and deployment algorithms," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, 2008. To appear.

[12] J. W. Jaromczyk and G. T. Toussaint, "Relative neighborhood graphs and their relatives," *Proceedings of the IEEE*, vol. 80, no. 9, pp. 1502–1517, 1992.

[13] J. Cortés, S. Martínez, and F. Bullo, "Spatially-distributed coverage optimization and control with limited-range interactions," *ESAIM. Control, Optimisation & Calculus of Variations*, vol. 11, pp. 691–719, 2005.

[14] L. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*. Cambridge University Press, second ed., 1997.