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Synchronization of Beads on a Ring

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Abstract— **This paper analyzes a discrete-time algorithm to synchronize an even number of agents moving clockwise and counterclockwise on a boundary. Each agent or "bead" changes direction upon encountering another bead moving in the opposite direction. Communication is sporadic: only when two beads come sufficiently close they are able to exchange information. We propose a novel algorithm based on the distributed computation of dominance regions and common speed, as well as, on a careful balancing of accelerate/decelerate strategies outside of dominance regions. Our theoretical analysis relies upon consensus algorithms tools and upon the assumption that initially half of all agents move clockwise and the other half move counterclockwise.**

I. INTRODUCTION

This work is motivated by applications of sensor networks that require the observation of certain spatial regions and, in particular, by surveillance tasks that can be simplified by the monitoring of the boundary of those regions. For example, this strategy can be employed for the monitoring of perimeters and of delimited areas and the tracking of targets within those areas. Coordination algorithms on boundaries can also be used for the tracking of evolving environmental phenomena such as spreading fires or chemical spills.

In devising such algorithms, several limitations arise. An important constraint is that of scalability. Each agent should be able to operate with limited information from others so that the performance of the algorithm is not compromised by an increased number of agents in the network. Due to the overheads in radio transmissions, frequent communications, even if the packets are small, is energy inefficient, see [1]. Therefore, it is preferable that agents communicate sporadically (perhaps, with longer messages) than frequently. Motivated by these issues, we propose and analyze a distributed algorithm that allows a mobile sensor network to monitor boundaries. The algorithm synchronizes a collection of n agents or beads, moving on a ring, so that each bead patrols a sector of the ring. An agent will meet, or impact, with the neighboring agents always at boundaries of its sector. The algorithm requires only occasional communication – two agents exchange information only when they impact.

Several cooperative algorithms have been proposed in boundary tracking problems. The paper [2] presents an algorithm to optimize the shape of a multi-vehicle formation to track level sets of environmental fields. The algorithm proposed in [3] for boundary tracking makes use of "elastic snakes" from the image processing literature. More relevant to this paper is the reference [4], which presents a synchronization algorithm for cooperative surveillance of a forest fire using a team of unmanned aerial vehicles. Other references on boundary tracking include [5], [6].

The dynamics of N-beads sliding on a frictionless ring has been subject of numerous papers; see for instance [7] and references therein. In particular in [7], the authors study extensively the case of $N = 3$ and prove the existence of periodic as well as chaotic orbits. The authors also describe how to use the three-bead system dynamics for a random number generator algorithm which is computationally efficient. In this paper we show that synchronization (a particular periodic orbit) can be achieved by modifying the impact law and making use of the theory of discrete-time consensus algorithms.

Consensus algorithms have been extensively studied; see [8], [9], [10] and references therein. Attractive properties of these algorithms are convergence under delays and communication failure, and robustness to communication noise. In particular we make use of discrete-time consensus algorithms and the analysis provided in [10] that guarantees convergence under mild assumptions on connectivity and general stochastic matrices. Other related papers to consensus algorithms and synchronization are those based on Kuramoto oscillators and cyclic pursuit; see [11], [12], [13] defined in both continuous and discrete time. A problem similar to the one we consider here is discussed in [14].

The contributions of this paper can be summarized as follows. We design a distributed algorithm to patrol a circular boundary by an even number of agents. The agents can be deployed with arbitrary initial positions and speeds. At the desired steady state, every agent patrols a sector of equal length, all agents move at the same speed, and neighboring agents meet always at the same point. Two agents exchange information only when they impact. We prove a local convergence result – the agents will reach the desired synchronized steady state – under the assumption that initially half of the agents move in the clockwise direction and the rest move in the counterclockwise direction. Extensive simulations show that synchronization is reached in general.

A similar problem is considered in [4], where pairs of agents have to be released at the same point, sequentially, and with the same speed. In contrast, in our algorithm the agents can be released at arbitrary positions, with arbitrary speeds and directions, as long as half of the agents move clockwise direction and the rest move in the counterclockwise direction.

The paper is organized as follows. Section II introduces notation employed and describes in detail what is meant

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by agent or bead synchronization on a circular boundary. The discrete-time Synchronization Algorithm is presented in Section III. The main results that allow to analyze the algorithm are included in IV. After this we present simulations in Section V showing that convergence of the algorithm is indeed possible in more general cases. Finally, we summarize the results in Section VI.

Notation

On the unit circle $\mathbb{S}^1$, by convention, let us define positions as angles measured counterclockwise from the positive horizontal axis. The *counterclockwise distance between two angles* $\mathrm{dist}_{cc}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to [0, 2\pi)$ is the path length from an angle to the other traveling counterclockwise. Specifically, if $x, y \in \mathbb{S}^1$, then $\mathrm{dist}_{cc}(x, y) = (y - x) \mod 2\pi$, where $x \mod 2\pi$ is the remainder of the division of x by 2π. We denote by $\mathbf{1} \in \mathbb{R}^{n\times 1}$ the column vector with entries all equal to 1.

II. MODEL AND PROBLEM STATEMENT

In this section we describe a synchronized collection of beads moving on a circle and our model of robotic agents.

Definition 1 (Synchronization): Consider a collection of n beads moving, with no friction, on a ring and suppose impacts among them are elastic. The collection of beads is *synchronized* if any two beads impact always at the same point, the time interval between two consecutive impacts has the same length, and all the beads impacts simultaneously. In other words, in a synchronized collection, each bead moves back and forth between the same two points whose counterclockwise distance is $2\pi/n$.

An example of a collection of four beads in sync is shown in Figure 1.

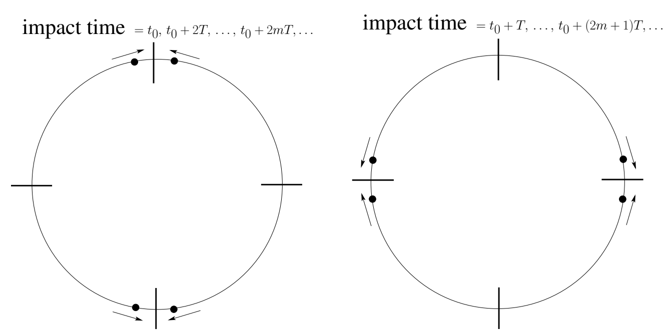

Fig. 1. The figure shows a collection of four beads which are synchronized.

In this paper we propose and analyze a distributed algorithm that will steer a collection of "intelligent beads," i.e., mobile robots, to be synchronized according to Definition 1.

The model of agent we consider is described as follows. We assume a collection of n agents moves on the unit circle $\mathbb{S}^1$. Let $p_i(t) \in [0, 2\pi)$, $i \in \{1, \ldots, n\}$ be the agents' positions at time $t \geq 0$, ordered in counterclockwise direction. Each agent knows its position on the circle. Each agent is equipped with a short-range communication device – we call a communication instant an "impact" because of the bead analogy. We use the identification $0 \equiv n$ and $n+1 \equiv 1$.

To achieve the synchrony as given in Definition 1, a necessary condition is that the number of beads in the collection is even.

III. SYNCHRONIZATION ALGORITHM

In this section we describe an algorithm that allows the collection of agents to achieve synchronization. We begin by defining all variables that each agent maintains in memory and we later state how these are updated as time evolves and "communication impacts" take place.

Let us define $d_i(t) \in \{-1, +1\}$ to be the direction of motion of the i-th bead, and let the counterclockwise direction of motion be positive. Let $\nu_i(t)$ be the i-th bead's nominal speed and let $v_i(t) \in \{d_i(t)\nu_i(t), d_i(t)f\nu_i(t), d_i(t)h\nu_i(t)\}$ be the actual velocity at which the i-th bead is moving at time t, with $f \in]0.5, 1[$ and $h = \frac{f}{2f-1} > 1$. The agents move in such as a way that their order never changes.

Definition 2 (Dominance region): Let $\mathcal{D}_i(t)$ be the dominance region of bead i at time $t \geq 0$, with $L_i(t)$ and $U_i(t)$ as its clockwise and counterclockwise boundary, then $\mathcal{D}_i(t) = \{\theta \in \mathbb{S}^1 \mid \mathrm{dist}_{cc}(L_i(t), \theta) \leq \mathrm{dist}_{cc}(L_i(t), U_i(t))\}$. Let $C_i(t)$ be the center of the dominance region, $C_i(t) = L_i(t) + \frac{1}{2}\,\mathrm{dist}_{cc}(L_i(t), U_i(t))$.

Definition 3 (Admissible initial conditions): The set of *admissible initial condition* $\mathcal{A}_{\text{ic}}$ is defined as follows:

- $\sum_{i=1}^{n} d_i(0) = 0$; in other words, at time $t = 0$, $n/2$ beads are moving clockwise and $n/2$ are moving counterclockwise,
- $\nu_i(0) > 0$, for all $i \in \{1, \ldots, n\}$, and
- $p_i(0) \neq p_j(0)$, for all $i \neq j \in \{1, \ldots, n\}$.

Definition 4 (Impacts classification): If at time t, $p_i(t) = p_{i+1}(t)$ then an impact has occurred between beads i and $i+1$. If $d_i(t) = d_{i+1}(t)$ then the impact is called "type head-tail impact" otherwise is called "type head-head impact."

We assume that each bead knows its position on the circle and is enabled with a short-range communication device.

It is convenient to denote by $x_i(t)$ the logic state that bead i maintains in its memory:

$$x_i(t) := \{p_i(t), \nu_i(t), d_i(t), U_i(t), L_i(t)\},$$

where $p_i(t)$ is the current position, $\nu_i(t)$ is the nominal speed, $d_i(t)$ is the direction of motion, and $U_i(t)$ and $L_i(t)$ are clockwise and counterclockwise boundaries of the dominance region. Furthermore, while the initial conditions $p_i(0)$, $\nu_i(0)$, and $d_i(0)$ belong to the set $\mathcal{A}_{\text{ic}}$ as in Definition 3, for the dominance region and its boundary we have $\mathcal{D}_i(0) = L_i(0) = U_i(0) = p_i(0)$.

If at time t an impact occurs involving beads i and $i+1$, for some $i \in \{1, \ldots, n\}$, then the two beads first calculate the center of their dominance region:

$$C_j(t) = L_j(t) + \frac{\mathrm{dist}_{cc}(L_j(t), U_j(t))}{2},\ j \in \{i, i+1\}.$$

Then, the beads involved in the impact update their logic states as follows:

$$U_i(t^+) = L_{i+1}(t^+) = C_i(t) + \frac{\mathrm{dist}_{cc}(C_i(t), C_{i+1}(t))}{2}, \quad (1)$$

$$\nu_i(t^+) = \nu_{i+1}(t^+) = \frac{\nu_i(t) + \nu_{i+1}(t)}{2}; \quad (2)$$

if the impact is of "type head-head," then

$$d_j(t^+) = -d_j(t),\ j \in \{i, i+1\}, \quad (3)$$

where the upper-script $+$ indicates the value of the state variables right after the impact. At all time $t \geq 0$ the actual velocity v_i is calculated as a function of the logic state $x_i(t)$:

$$v_i(x_i(t)) = \begin{cases} d_i(t)\nu_i(t), & \text{if } p_i(t) \in \mathcal{D}_i(t), \\ d_i(t) f \nu_i(t), & \text{if } p_i(t) \notin \mathcal{D}_i(t) \text{ and } i \\ & \quad \text{is moving away from it,} \\ d_i(t) h \nu_i(t), & \text{if } p_i(t) \notin \mathcal{D}_i(t) \text{ and } i \\ & \quad \text{is moving towards it,} \end{cases} \quad (4)$$

where, we recall, $f \in]0.5, 1[$ and $h = \frac{f}{2f-1} > 1$. For simplicity of notation we will often use $v_i(t)$ instead of $v_i(x_i(t))$. At time $t = 0$ the actual velocity is $v_i(0) = d_i(0)\nu_i(0)$ and its absolute value will not change until the first impact occurs.

IV. Convergence analysis

Let us now construct an undirected graph $\mathcal{G}(t)$ with vertex set $\{1, \dots, n\}$ and edge from i to $i+1$ if the beads i and $i+1$ collide at time t. To prove the correctness of the SYNCHRONIZATION ALGORITHM we need to show the following result.

Proposition 5 (Uniform connectivity): Along the trajectories of the closed loop system induced by the SYNCHRONIZATION ALGORITHM, with $(x_1(0), \dots, x_n(0)) \in \mathcal{A}_{ic}$, for all $t_0 \geq 0$ the graph $\bigcup_{t \in [t_0, t_0 + 2\pi/(f\nu_{\min})]} \mathcal{G}(t)$ is connected.

The proof of Proposition 5 builds up on the following facts.

Lemma 6 (Properties): Along the trajectories of the SYNCHRONIZATION ALGORITHM, with $(x_1(0), \dots, x_n(0)) \in \mathcal{A}_{ic}$:

- (i) $\sum_{i=1}^n d_i(t) = 0$, that is, at any instant of time $n/2$ beads are moving clockwise and $n/2$ are moving counterclockwise,
- (ii) any two dominance regions are disjoint sets or at most share a boundary point, furthermore their label index increases in the counterclockwise direction,
- (iii) the order of the beads is preserved, i.e., for all $i \in \{1, \dots, n\}$, $t \geq 0$, and for $j \neq i$, $\mathrm{dist}_{cc}((p_{i-1}(t), p_i(t)) \leq \mathrm{dist}_{cc}(p_{i-1}(t), p_{i+1}(t))$ and $\mathrm{dist}_{cc}((p_{i-1}(t), p_j(t)) \geq \mathrm{dist}_{cc}(p_{i-1}(t), p_{i+1}(t))$. Therefore, a bead i can be involved only in impacts with its immediate neighbors $i-1$ and $i+1$.

Proof: For the interest of brevity we omit the proof of this result. ■

Lemma 7 (Impacts in bounded interval): Let $\nu_{\min} = \min\{\nu_i(0) |\ i \in \{1, \dots, n\}\}$. Along the trajectories of the closed-loop system induced by the SYNCHRONIZATION ALGORITHM, with $(x_1(0), \dots, x_n(0)) \in \mathcal{A}_{ic}$, for all $i \in \{1, \dots, n\}$ and for all $t_0 > 0$, bead i will impact at least once with both its neighbors $i-1$ and $i+1$ across the interval $[t_0, t_0 + \frac{2\pi}{f\nu_{\min}}]$.

Proof: Note that $\min_{i \in \{1,\dots,n\}} \nu_i(t) \geq \min_{i \in \{1,\dots,n\}} \nu_i(0) = \nu_{\min}$ because of equation (2). Therefore for any $t > 0$ the lowest possible speed at which a bead can travel is $f\nu_{\min}$. We first show that at most after $\frac{\pi}{f\nu_{\min}}$ any bead will have a "type head-head" impact with one of its neighbors.

First, any bead i can only impact with neighbors $i+1$ and $i-1$ because of Lemma 6, part (iii). The necessary time for two beads i, $i+1$ to impact depends on their positions, the directions of motion and the speeds they are traveling with.

In the worst possible case at a time $t = t_0$ all the beads are clustered in a small arc of $\mathbb{S}^1$ of length ϵ, with i and $i+1$ at the opposite ends of the arc (i.e., $\mathrm{dist}_{cc}(p_{i+1}(t_0), p_i(t_0)) = \epsilon$), $d_i(t_0) = d_{i+1}(t_0)$, and the speeds have the smallest possible value $|v_i(t_0)| = |v_{i+1}(t_0)| = f\nu_{\min}$.

Let us suppose $d_i(t_0) = d_{i+1}(t_0) = +1$. That is, $i+1$ is moving towards the cluster of beads and i is moving away from it. Because of Lemma 6, part (i), we have that $\sum_{i=1}^n d_i(t_0) = 0$ and this implies that $i+1$ can travel at most for $\frac{\epsilon}{2f\nu_{\min}}$ before having a "type head-head" impact. So at $t_1 \leq t_0 + \frac{\epsilon}{2f\nu_{\min}}$, $d_{i+1}(t_1) = -1$, and $\mathrm{dist}_{cc}(p_{i+1}(t_1), p_i(t_1)) \geq \epsilon$. This is true because by assumption $|v_i(t_0)| = |v_{i+1}(t_0)|$ and i could have had a "type head-tail" impact with $i-1$ so that $|v_i(t_1)| \geq f\nu_{\min}$. Now, suppose that even after the impact $|v_{i+1}(t_1)| = f\nu_{\min}$, then beads i and $i+1$ are moving towards each other and $\mathrm{dist}_{cc}(p_i(t_1), p_{i+1}(t_1)) \leq 2\pi - \epsilon$. They will then meet at time $t_2 \leq t_1 + \frac{2\pi-\epsilon}{2f\nu_{\min}} \leq t_0 + \frac{\epsilon}{2f\nu_{\min}} + \frac{2\pi-\epsilon}{2f\nu_{\min}} = t_0 + \frac{\pi}{f\nu_{\min}}$.

After the impact with $i+1$, $d_i(t_2) = -1$ and, therefore, in its next "type head-head" impact bead i will meet $i-1$. Following the same reasoning, we have that at most after $\frac{\pi}{f\nu_{\min}}$ the two beads i and $i-1$ will meet. Hence across the interval $[t_0, t_0 + \frac{2\pi}{f\nu_{\min}}]$ any bead will impact at least once with both its neighbors. ■

Proof: [of Proposition 5] Because of Lemma 7, for all i and for all t_0 there exist t_1 and $t_2 \in [t_0, t_0 + \frac{2\pi}{f\nu_{\min}}]$ such that $\mathcal{G}(t_1)$ and $\mathcal{G}(t_2)$ have respectively an edge between vertices i and $i+1$ and between vertices i and $i-1$. Then, clearly the graph

$$\bigcup_{t \in [t_0, t_0 + 2\pi/(f\overline{\nu}(0))]} \mathcal{G}(t)$$

is connected. ■

Lemma 8 (Velocity convergence): Let $\nu(t) = [\nu_1(t), \dots, \nu_n(t)]^T \in \mathbb{R}^{n\times 1}$. Along the trajectories of the closed-loop system induced by the SYNCHRONIZATION ALGORITHM, with $(x_1(0), \dots, x_n(0)) \in \mathcal{A}_{ic}$:

$$\lim_{t\to+\infty} \left\| \nu(t) - \frac{\mathbf{1}^T \nu(0)}{n} \mathbf{1} \right\| = 0\,.$$

Proof: For all $i \in \{1, \dots, n\}$, define $A_i \in \mathbb{R}^{n \times n}$ by:

$$[A_i]_{lm} = \begin{cases} \frac{1}{2}, & \text{if } l = m = i \text{ or } l = m = i+1, \\ \frac{1}{2}, & \text{if } (l,m) \in \{(i,i+1),(i+1,i)\}, \\ \delta_{lm}, & \text{otherwise}. \end{cases}$$

Because of equation (2), if at time t an impact between i and $i+1$ occurs:

$$\nu(t^+) = A_i \nu(t).$$

Therefore the dynamics of $\nu(t)$ is just the average consensus dynamics with matrices A_i and, because of Proposition 5, the consensus is asymptotically reached (see [10]). Clearly, because A_i, $i \in \{1, \dots, n\}$, are doubly stochastic, the consensus value is $\frac{1}{n}\sum_{i=1}^{n} \nu_i(0)$. ■

Lemma 9 (Dominance region convergence): Let $\ell_i(t) = \mathrm{dist}_{\mathrm{cc}}(L_i(t), U_i(t))$ be the length of the dominance region $\mathcal{D}_i(t)$ for $i \in \{1, \dots, n\}$, and $\ell(t) = [\ell_1(t), \dots, \ell_n(t)]^T \in \mathbb{R}^{n\times 1}$. Along the trajectories of the closed-loop system induced by the SYNCHRONIZATION ALGORITHM, with $(x_1(0), \dots, x_n(0)) \in \mathcal{A}_{\mathrm{ic}}$:

$$\lim_{t\to+\infty} \big\| \ell(t) - \frac{\mathbf{1}^T \ell(t)}{n}\mathbf{1} \big\| = 0\,.$$

Proof: From equation (1) we have that after the impact between i and $i+1$:

$$\ell_i(t^+) = \frac{3}{4}\ell_i(t) + \frac{1}{4}\ell_{i+1}(t),$$
$$\ell_{i+1}(t^+) = \frac{1}{4}\ell_i(t) + \frac{3}{4}\ell_{i+1}(t).$$

Now, for $i \in \{1, \dots, n\}$, define $B_i \in \mathbb{R}^{n\times n}$ by:

$$[B_i]_{lm} = \begin{cases} \frac{3}{4}, & \text{if } l = m = i \text{ or } l = m = i+1, \\ \frac{1}{4}, & \text{if } (l,m) \in \{(i,i+1),(i+1,i)\}, \\ \delta_{lm}, & \text{otherwise}. \end{cases}$$

Then, if at time t an impact between i and $i+1$ occurs, the dynamics for $\ell(t)$ is simply:

$$\ell(t^+) = B_i \ell(t).$$

Once again, the dynamics of $\ell(t)$ is just the weighted average consensus dynamics with matrices B_i and, because of Proposition 5, the consensus is asymptotically reached (see [10]). Since $\sum_{i=1}^{n} \ell_i(t) = 2\pi$, or equivalently because B_i, $i \in \{1, \dots, n\}$, are doubly stochastic, we have that $\ell_i(t) \to \frac{2\pi}{n}$ asymptotically. ■

We have then proved that asymptotically the nominal velocities $\nu_i(t)$ will be equal to the average of the initial nominal velocities and the lengths of the dominance regions $\mathcal{D}_i(t)$ will asymptotically be equal to $2\pi/n$. We will now prove that the SYNCHRONIZATION ALGORITHM will steer the collection of beads to be in sync for a set of initial conditions smaller than $\mathcal{A}_{\mathrm{ic}}$.

Theorem 10 (Convergence to Synchrony): For all $i \in \{1, \dots, n\}$, let $\nu_i(0) = \overline{\nu} > 0$, let $\mathrm{dist}_{\mathrm{cc}}(L_i(0), L_{i+1}(0)) = \frac{2\pi}{n}$, and let $d_i(0) = -d_j(0)$ for $j \in \{i-1, i+1\}$. Let $\gamma_i = \mathrm{dist}_{\mathrm{cc}}(C_i(0), p_i(0))$, let $\delta_i = \min\{\gamma_i, 2\pi - \gamma_i\}$, and let $\delta = [\delta_1, \dots, \delta_n]^T \in \mathbb{R}^{n\times 1}$. Let T_i^k be the instant in which bead i passed by the center of its dominance region for the k-th time and $T^k = [T_1^k, \dots, T_n^k]^T \in \mathbb{R}^{n\times 1}$. If $\|\delta - \frac{\mathbf{1}^T\delta}{n}\mathbf{1}\|$ is sufficiently small, then:

$$\lim_{k\to+\infty} \big\| T^k - \frac{\mathbf{1}^T T^k}{n}\mathbf{1} \big\| = 0\,.$$

Proof: Before tackling the proof it is useful to see that both the quantities $\|\delta - \frac{\mathbf{1}^T\delta}{n}\mathbf{1}\|$ and $\|T^k - \frac{\mathbf{1}^T T^k}{n}\mathbf{1}\|$ are measures of the asynchrony of the collection of beads. However, due to the switching nature of the dynamics of the beads, the asymptotic behavior of T^k is more simple to analyze. On the other hand δ is a more suitable quantity to describe the asynchrony at time 0.

Let us suppose that at time t the beads i and $i+1$, with directions $d_i(t) = -d_{i+1}(t) = +1$, are about to collide. We know that T_i^k and T_{i+1}^k, for some k, are the times at which they passed by the centers of their dominance regions. If $T_i^k < T_{i+1}^k$, that is bead i is *early* with respect to bead $i+1$, the impact will occur in $\mathcal{D}_{i+1}$ as shown in Figure 2, otherwise it will occur in $\mathcal{D}_i$. Without loss of generality we suppose that the impact will occur in $\mathcal{D}_{i+1}$.

Let $\eta = (T_{i+1}^k - T_i^k)\overline{\nu}$. At $t_0 = T_i^k + \frac{(\pi/n)}{\overline{\nu}}$ bead i reaches the boundary of its dominance region (i.e., $p_i(t_0) = U_i$), and $\mathrm{dist}_{\mathrm{cc}}(p_i(t_0), p_{i+1}(t_0)) = \eta$. This is true because when traveling inside its dominance region $v_i(t) = d_i(t)\nu_i(t)$, and by assumption $\nu_i(0) = \overline{\nu}$ for all i and, therefore, for all $t \geq 0$. Let t_1 be the time at which the two beads collide and let $\mu = \mathrm{dist}_{\mathrm{cc}}(U_i, p_i(t_1))$. Then we have that:

$$\eta + v_{i+1}(t_1 - t_0) = v_i(t_1 - t_0),$$
$$\mu = v_i(t_1 - t_0).$$

Note that the speed for bead i is $f\overline{\nu}$ because it is mowing away from its dominance region , while for $i+1$ is $\overline{\nu}$, therefore:

$$\eta - \overline{\nu}(t_1 - t_0) = f\overline{\nu}(t_1 - t_0),$$
$$\mu = f\overline{\nu}(t_1 - t_0).$$

Solving for μ we have:

$$\mu = \eta\frac{f}{1+f} = (T_{i+1}^k - T_i^k)\overline{\nu}\frac{f}{1+f}. \tag{5}$$

After the impact the directions of both beads change because the impact is of "type head-head," hence bead i is rushing back to its cell with speed $h\overline{\nu}$. Let t_2 be the time in which bead i crosses the boundary of its cell, i.e., $p_i(t_2) = U_i$, then:

$$t_2 - t_0 = t_2 - t_1 + t_1 - t_0 = \frac{\mu}{h\overline{\nu}} + \frac{\mu}{f\overline{\nu}} = 2\frac{\mu}{\overline{\nu}},$$

because $h = \frac{f}{2f-1}$. Let us calculate T_i^{k+1} and T_{i+1}^{k+1}:

$$T_i^{k+1} = T_i^k + \frac{2}{\overline{\nu}}(\frac{\pi}{n} + \mu), \tag{6}$$

$$T_{i+1}^{k+1} = T_{i+1}^k + \frac{2}{\overline{\nu}}(\frac{\pi}{n} - \mu). \tag{7}$$

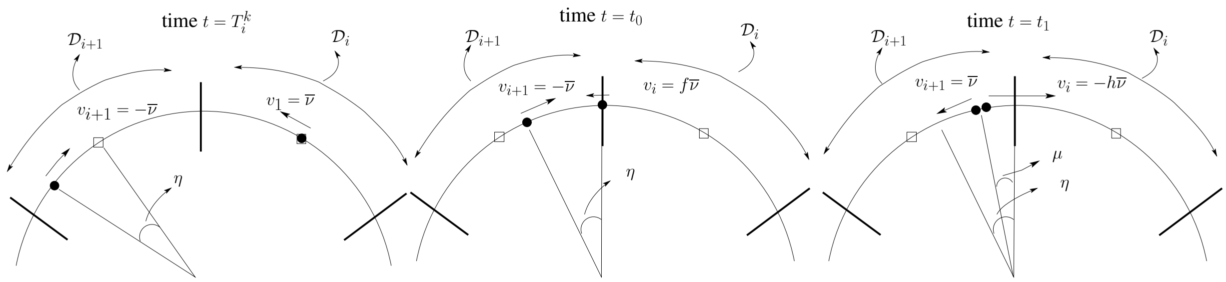

Fig. 2. The figure shown positions, black circles, and velocities, of beads i and $i+1$ at time $t = T_i^k$, $t = t_0$ and $t = t_1$ as described in Theorem 10. The squares are the centers C_i and C_{i+1} of the dominance regions $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$.

Substituting (5) in (6) and in (7):

$$T_i^{k+1} = \frac{1-f}{1+f} T_i^k + \frac{2f}{1+f} T_{i+1}^k + \frac{2\pi}{n\overline{\nu}},$$
$$T_{i+1}^{k+1} = \frac{2f}{1+f} T_i^k + \frac{1-f}{1+f} T_{i+1}^k + \frac{2\pi}{n\overline{\nu}}.$$

Note that $0 < \frac{1-f}{1+f} < 1/3$ and $2/3 < \frac{2f}{1+f} < 1$ since $f \in]0.5, 1[$. Now, let us define the matrices C^{even} and $C^{\text{odd}} \in \mathbb{R}^{n \times n}$ by

$$[C^{\text{even}}]_{lm} = \begin{cases} \frac{1-f}{1+f}, & \text{if } l = m, \\ \frac{2f}{1+f}, & \text{if } (l,m) \in \{(i,i+1),(i+1,i)\},\ i \text{ even,} \end{cases}$$

and by

$$[C^{\text{odd}}]_{lm} = \begin{cases} \frac{1-f}{1+f}, & \text{if } l = m, \\ \frac{2f}{1+f}, & \text{if } (l,m) \in \{(i,i+1),(i+1,i)\},\ i \text{ odd.} \end{cases}$$

Then, if the first impact after $t = 0$ is between i and $i+1$, and i is odd the vector T^k evolves as follows:

$$T^{k+1} = \begin{cases} C^{\text{odd}} T^k + \frac{2\pi}{n\overline{\nu}} \mathbf{1}, & \text{if } k \text{ odd,} \\ C^{\text{even}} T^k + \frac{2\pi}{n\overline{\nu}} \mathbf{1}, & \text{if } k \text{ even.} \end{cases} \tag{8}$$

If the first impact is between i and $i+1$, and i is even, equation (8) is still valid as long as the definitions of C^{odd} and C^{even} are exchanged. In any case, the dynamics of T^k is just the weighted average consensus dynamics with matrices C^{odd} and C^{even}, and, because of Proposition 5, the consensus is asymptotically reached (see [10]). ■

Although Theorem 10 proves convergence to synchronization only locally, simulations show that indeed the set of initial conditions for which the SYNCHRONIZATION ALGORITHM allows a collection of beads to reach synchronization is quite large. In the next remark we give some insight.

Remark 11: The SYNCHRONIZATION ALGORITHM leads to a dynamical system that can be seen as a cascade of three dynamical systems: the dynamical systems of the nominal velocities $\nu_i(t)$, the dynamical systems of the dominance regions $\mathcal{D}_i(t)$, and the dynamical system of the synchrony T_i^k. The dynamical systems of the nominal velocities and of the dominance regions are independent from each other and independent from the dynamics of the synchrony, furthermore they act as disturbances on the latter. As proved in Lemma 8 and Lemma 9, $\lim_{t\to+\infty} \|\nu(t) - \frac{\mathbf{1}^T \nu(t)}{n}\mathbf{1}\| = 0$ and $\lim_{t\to+\infty} \|\ell(t) - \frac{\mathbf{1}^T \ell(t)}{n}\mathbf{1}\| = 0$ for all initial conditions in $\mathcal{A}_{\text{ic}}$ – the consensus of the nominal speeds and of the lengths of the dominance regions is guaranteed. Furthermore, since the convergence is uniform and the dynamics are linear the convergence is exponential. For the same reasons the convergence of $\|T^k - \frac{\mathbf{1}^T T^k}{n}\mathbf{1}\|$ is exponential. Next, if the inputs $\|\nu(t) - \frac{\mathbf{1}^T \nu(t)}{n}\mathbf{1}\|$ and $\|\ell(t) - \frac{\mathbf{1}^T \ell(t)}{n}\mathbf{1}\|$ enters linearly in the dynamics of T_i^k, then the local stability properties of the equilibrium $\|T^k - \frac{\mathbf{1}^T T^k}{n}\mathbf{1}\| = 0$ are not destroyed. This follows from Input-to-State Stability of exponentially stable systems [15]. If this holds, then the restrictive assumptions for Theorem 10 are that $\|\delta - \frac{\mathbf{1}^T \delta}{n}\mathbf{1}\|$ is sufficiently small and that $d_i(0) = -d_j(0)$ for $j \in \{i-1, i+1\}$, while the assumptions that $\nu_i(0)$, $i \in \{1,\dots,n\}$, have the same value and that $\text{dist}_{\text{cc}}(L_i(0), L_{i+1}(0)) = 2\pi/n$ are not restrictive.•

V. SIMULATIONS

In this section we presents simulation results obtained by implementing the SYNCHRONIZATION ALGORITHM with $n = 8$ beads. We assume that p_i, for $i \in \{1,\dots,n\}$, are randomly positioned on $\mathbb{S}^1$, and that $\nu_i(0)$, for $i \in \{1,\dots,n\}$, are uniformly distributed in $]0,1]$. Finally, we set $d_1(0) = d_2(0) = d_4(0) = d_6(0) = +1$ and $f = 0.7$.

Figure 3 shows the positions of the eight beads vs time. Clearly, asymptotically each bead meets its neighbor at the same location on the circle, reaching synchrony.

Figure 4 shows $\max_i \nu_i(t) - \min_i \nu_i(t)$, which is a measure of disagreement of the nominal speeds. As expected the disagreement goes to zero asymptotically.

Figure 5 shows the positions and the dominance region boundaries for bead $i = 5$. The solid line represents $p_5(t)$, the dash-dot line represents $L_i(t)$, and the thicker solid line represents $U_i(t)$. The distance $\text{dist}_{\text{cc}}(L_i(t), U_i(t))$ asymptotically approaches $360/n = 45$ degrees.

VI. CONCLUSIONS

We presented and analyzed an algorithm that synchronizes a collection of n agents or beads, moving on a ring, so that each bead patrols only a sector of the ring. The algorithm is distributed and requires that two agents exchange information only when they meet. We proved that the proposed algorithm allows the agents to reach the desired steady state for certain initial conditions. Simulations show convergence to the desired steady state for a larger set of initial conditions. Motivated by the implementation results, we plan to look

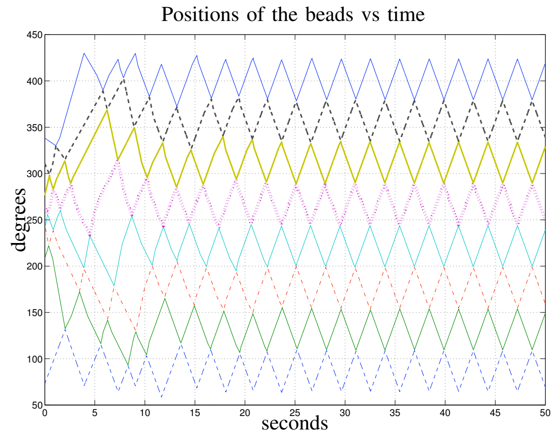

Fig. 3. This figure shows p_i vs time, obtained by implementing the SYNCHRONIZATION ALGORITHM with $n = 8$ beads. The positions of the beads $2, 4, 6, 8$ are represented by solid lines, while the dash line, dash-dot line, point line, and thicker dash line represent the positions of beads $1, 3, 5, 7$.

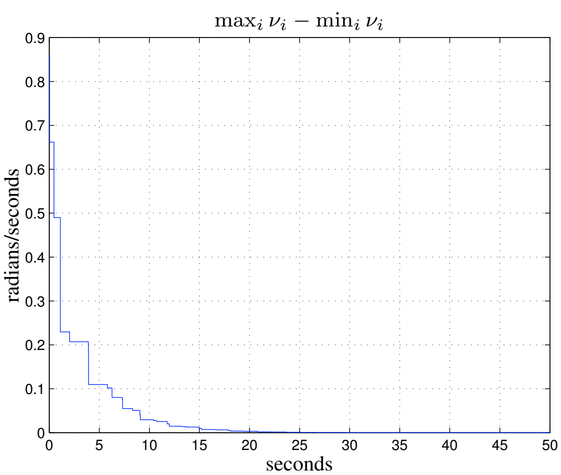

Fig. 4. This figure shows $\max_i \nu_i - \min_i \nu_i$ vs time, obtained by implementing the SYNCHRONIZATION ALGORITHM with $n = 8$ beads.

for a different measure of the asynchrony which may be more suitable to prove convergence for a larger set of initial conditions.

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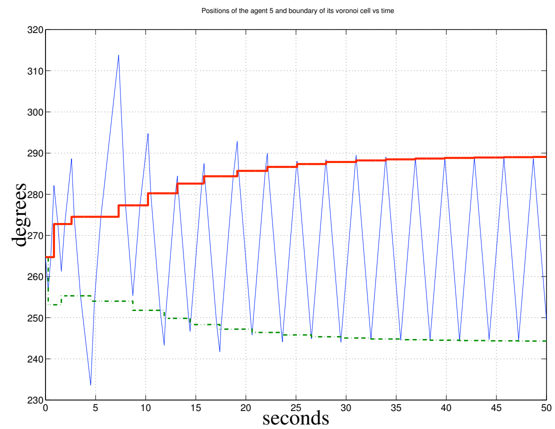

Fig. 5. This figure shows $p_5(t)$ (solid line), $U_5(t)$ (ticker solid line), and $L_5(t)$ (dash-dot line), obtained by implementing the SYNCHRONIZATION ALGORITHM with $n = 8$ beads.

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