Practical multiagent rendezvous through modified circumcenter algorithms

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Abstract

We present a class of modified circumcenter algorithms that allow a group of agents to achieve "practical" rendezvous when they are only able to take noisy measurements of neighbors. Assuming a uniform detection probability in a disk about each neighbor true position, we show how initially connected agents converge to a practical stability ball. More precisely, a deterministic analysis allows to guarantee convergence to such a ball under r-disk graph connectivity in 1D and with a connectivity-to-noise ratio of $r/\sigma > 7$. A stochastic analysis leads to a similar convergence result in probability, but for any $r/\sigma > 1$, and under a sequence of switching graphs connecting agents in 2D every T steps. We include several simulations to discuss the performance of the proposed algorithms.

Key words: multiagent systems, robotic networks, motion coordination, rendezvous algorithm

1 Introduction

The topic of distributed algorithms for robotic networks is attracting an intense research activity in the last years, see e.g. [1]. As a consequence of this, a wealth of algorithms is being proposed together with novel analysis tools to evaluate their performance. Clearly, an important aspect to consider is that of robustness. If possible, a characterization of what typical degraded behaviors are and how those are affected by the network size should be provided. When not satisfactory, the performed analysis may help find an alternative solution.

Motivated by this, we discuss how the nonlinear Circumcenter Algorithm, see [2], can be made robust with respect to measurement noise. This complements the work in [2], which observed good performance of the algorithm in simulation, and the work in [3–5], which respectively considered asynchronous versions of the algorithm, and proven convergence under sequences of switching graphs. Other related papers include [6–8], which study how consensus algorithms are robust to communication, measurement noise and quantization errors. However, the type of algorithms considered in these works are linear, while the Circumcenter Algorithm is nonlinear and agents' motion is constrained.

The contributions of this paper can be summarized as follows. First, we propose an alternative to the standard Circumcenter Algorithm that allows agents to maintain connectivity and rendezvous without explicitly accounting for motion constraints. Assuming that agents are able to measure neighbors within radius σ about their true positions, we present two possible modifications of the algorithms. The first one restricts further each agent's motion constraint set to guarantee connectivity of the network. In the second version, agents filter measurements of neighbors to make sure that they are still within the connectivity radius r.

We prove that the implementation of the algorithms in 1D using the r-disk graph for connectivity achieves rendezvous to a ball of diamater 2σ. This is a type

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of deterministic Input-to-state Stability result that, for the standard Circumcenter Algorithm, requires $r > 7\sigma$. As shown in simulations, the same type of bounded behavior does not hold for other graphs. In those cases, a stochastic analysis can be used to show a decreasing trend of the network diameter with probability one until agents are fully connected under the r-disk graph. We extend the result for any sequence of graphs over agents moving in 2D, satisfying certain connectivity assumption, and with a connectivity-to-noise ratio of $r/\sigma > 1$. Simulations also show that agents reach a practical stability ball with diameter much smaller than 2σ for the modified circumcenter algorithms when using the r-disk graph. With respect to [9], here we present the alternative to the Circumcenter Algorithm and extend the stochastic analysis of the algorithms to 2D.

The paper is organized as follows. Section 2 introduces premilinary notions, circumcenter algorithms and modifications. Section 3 includes a deterministic analysis of the modified circumcenter algorithms when implemented in 1D and over the r-disk graph. Section 4 includes a stochastic analysis of the algorithms in 2D. Finally, Section 5 illustrates the peformance of the algorithms in simulations and Section 6 presents some concluding remarks.

2 Preliminaries

Here, we review some notation for standard geometric objects; for additional information we refer the reader to [10]. We then recall the circumcenter and parallel circumcenter algorithms as discussed in [2,11,12]. The section concludes introducing the new class of modified circumcenter algorithms.

2.1 Basic geometric notions and notation

In what follows, $\mathbb{R}^d$ will refer to either $\mathbb{R}$ or $\mathbb{R}^2$. For a bounded set $S \subset \mathbb{R}^d$, we let $\mathrm{co}(S)$ denote the convex hull of S. For $p, q \in \mathbb{R}^d$, we let $(p, q) = \{\lambda p + (1-\lambda)q \mid \lambda \in (0,1)\}$ and $[p, q] = \mathrm{co}(\{p, q\})$ denote the *open* and *closed segment* with extreme points p and q, respectively. For a bounded set $S \subset \mathbb{R}^d$, we let $\mathrm{CC}(S)$ and $\mathrm{CR}(S)$ denote the *circumcenter* and *circumradius* of S, respectively, that is, the center and radius of the smallest-radius d-sphere enclosing S. The computation of the circumcenter and circumradius of a bounded set is a strictly convex problem and in particular a quadratically constrained linear program. For $p \in \mathbb{R}^d$, $B(p, r)$ and $D(p, r)$ denote the *open* and *closed disk* of center p and radius $r \in \mathbb{R}_{>0}$, respectively. Similarly, $\mathbb{S}(p, r)$ will denote the sphere of center p and radius $r \in \mathbb{R}_{>0}$. An arc on $\mathbb{S}(p, r)$ transversed counterclockwise from q_1 to q_2 will be denoted by $\mathrm{arc}_{\mathbb{S}(r,p)}(q_1, q_2)$. Here, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ will denote the positive and the nonnegative real numbers, respectively. Given points $q_1, q_2, q_3 \in \mathbb{R}^2$, we denote the angle at q_2 with positive orientation and formed by the vectors $q_2 - q_1$ and $q_3 - q_2$ as $\angle(q_1, q_2, q_3)$.

Let $\mathbb{F}(\mathbb{R}^d)$ be the collection of finite point sets in $\mathbb{R}^d$; we shall denote an element of $\mathbb{F}(\mathbb{R}^d)$ by $\mathcal{P} = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$, where $p_1, \dots, p_n$ are distinct points in $\mathbb{R}^d$. Let $\mathbb{G}(\mathbb{R}^d)$ be the set of undirected graphs whose vertex set is an element of $\mathbb{F}(\mathbb{R}^d)$. A *proximity graph function* $\mathcal{G}\colon \mathbb{F}(\mathbb{R}^d) \to \mathbb{G}(\mathbb{R}^d)$ associates to a point set $\mathcal{P}$ an undirected graph with vertex set $\mathcal{P}$ and edge set $\mathcal{E}_{\mathcal{G}}(\mathcal{P})$, where $\mathcal{E}_{\mathcal{G}}\colon \mathbb{F}(\mathbb{R}^d) \to \mathbb{F}(\mathbb{R}^d \times \mathbb{R}^d)$ has the property that $\mathcal{E}_{\mathcal{G}}(\mathcal{P}) \subseteq \mathcal{P} \times \mathcal{P} \setminus \mathrm{diag}(\mathcal{P} \times \mathcal{P})$ for any $\mathcal{P}$. Here, $\mathrm{diag}(\mathcal{P} \times \mathcal{P}) = \{(p, p) \in \mathcal{P} \times \mathcal{P} \mid p \in \mathcal{P}\}$. In other words, the edge set of a proximity graph depends on the location of its vertices. General properties of proximity graphs, basics on graph theory and examples can be found in [10,13,14]. In particular, we will make use of the r-disk proximity graph $\mathcal{G}_{\mathrm{disk}}(r)$, for $r \in \mathbb{R}_{>0}$, and over a set of vertices $\mathcal{P}$. In this graph, two agents $p_i, p_j \in \mathcal{P}$ are neighbors iff $\|p_i - p_j\| \leq r$. We denote the set of neighbors of agent $p_i \in \mathcal{P}$ in a proximity graph by:

$$\mathcal{N}_i(\mathcal{G}) = \{j \in \{1, \dots, n\} \mid (p_i, p_j) \in \mathcal{E}_{\mathcal{G}}(\mathcal{P})\},$$

and the cardinality of $\mathcal{N}_i(\mathcal{G})$ will be denoted as $n_i = |\mathcal{N}_i(\mathcal{G})|$. A sequence of finite point sets $\{\mathcal{P}(t) \mid t \in \mathbb{N} \cup \{0\}\}$, induces a sequence of graphs that we denote as $\mathcal{G}(t)$, when it is clear from the context that $\mathcal{G}(t) \equiv \mathcal{G}(\mathcal{P}(t))$, $t \in \mathbb{N} \cup \{0\}$.

For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda(q_0, q_1, Q)$ denote the solution of the strictly convex problem:

$$\begin{aligned} &\text{maximize } \lambda \\ &\text{subject to } \lambda \leq 1,\ (1-\lambda)q_0 + \lambda q_1 \in Q. \end{aligned} \tag{1}$$

Note that this convex optimization problem has the following interpretation: move along the segment from q_0 to q_1 the maximum possible distance while remaining in Q. Under the stated assumptions the solution exists and is unique.

2.2 Circumcenter algorithms

The following is an informal description of the Circumcenter Algorithm defined for a proximity graph $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$, with $r \in \mathbb{R}_{>0}$.

> *(Standard) Circumcenter Algorithm [2,4,5].* Each agent performs: (i) it detects its neighbors according to $\mathcal{G}$; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself; (iii) it moves toward this circumcenter while maintaining connectivity with its neighbors.

The asynchronous behavior of the algorithm for was studied in [3,4]. In [12], it was proven that, when implemented over a 1D space, it is not necessary to enforce the connectivity constraint. In other words, step (iii) can be rephased as "(iii) agent moves to the circumcenter of neighbors". Assuming that agents have knowledge of a frame with common orientation, the 1D algorithm can be extended to arbitrary dimensions as follows.

> *Parallel Circumcenter Algorithm [12].* Each agent performs: (i) it detects its neighbors according to $\mathcal{G}$; (ii) it projects the detected positions to each axis of its frame; (iii) it computes the circumcenters of each of the projected sets of positions on each axis; (iii) it moves to the point whose coordinates are given by each of those circumcenters.

For formal descriptions of these algorithms written in pseudocode we refer the reader to [5,12]. Yet there is another way to define an alternative circumcenter algorithm that does not require the explicit use of constraint sets. It is based on the following lemma.

Lemma 1 *Let $S \subset \mathbb{R}^2$ be the convex hull of a set of points $p_1, \ldots, p_n$. Let $s > 0$ and define the disks $D(p_i, s)$, $i \in \{1, \ldots, n\}$. If the intersection $\bigcap_{i=1}^n D(p_i, s)$ is non empty, then $\mathrm{CC}(S) \in \bigcap_{i=1}^n D(p_i, s)$.*

Proof. Let $p \in \bigcap_{i=1}^n D(p_i, s)$. By definition, we have

$$\|p - p_i\| \leq s, \qquad \forall\, i \in \{1, \ldots, n\}\,.$$

By the minimality property of $\mathrm{CR}(S)$, this implies $\mathrm{CR}(S) \leq s$. But then,

$$\|\,\mathrm{CC}(S) - p_i\| \leq \mathrm{CR}(S) \leq s, \qquad \forall\, i \in \{1, \ldots, n\},$$

which implies $\mathrm{CC}(S) \in \bigcap_{i=1}^n D(p_i, s)$. ■

Now the (standard) Circumcenter Algorithm can be replaced by the following algorithm.

> 1/2 *Circumcenter Algorithm.* Each agent performs: (i) it detects its neighbors according to $\mathcal{G}$; (ii) it computes the circumcenter of points comprised of itself and the midpoints between each neighbor and itself; (iii) it moves to the point whose coordinates are given by this circumcenter.

In other words, the new circumcenter by agent i becomes $\mathrm{CC}(\{p_i, \frac{p_i+p_j}{2} \,|\, j \in \mathcal{N}_i(\mathcal{G})\})$. Since $\|p_i - p_j\| \leq r$, then $\|p_i - \frac{p_i+p_j}{2}\| \leq \frac{r}{2}$, for all $j \in \mathcal{N}_i(\mathcal{G}) \cup \{i\}$. Therefore $p_i \in D(p_i, \frac{r}{2}) \cap \bigcap_{j\in\mathcal{N}_i(\mathcal{G})} D(\frac{p_i+p_j}{2}, \frac{r}{2}) \neq \emptyset$, and by Lemma 1 we have that $\mathrm{CC}(\{p_i, \frac{p_i+p_j}{2} \,|\, j \in \mathcal{N}_i(\mathcal{G})\}) \in D(p_i, \frac{r}{2}) \cap \bigcap_{j\in\mathcal{N}_i(\mathcal{G})} D(\frac{p_i+p_j}{2}, \frac{r}{2})$. That is, moving to this new circumcenter, we guarantee that agents will not lose connectivity with neighbors. At the same time, $\mathrm{CC}(\{p_i, \frac{p_i+p_j}{2} \,|\, j \in \mathcal{N}_i(\mathcal{G})\}) \in \mathrm{co}(\{p_i, p_j \,|\, j \in \mathcal{N}_i(\mathcal{G})\}) \setminus \{p_i, p_j \,|\, j \in \mathcal{N}_i(\mathcal{G})\}$; then, similarly to [5], the diameter can be seen to decrease strictly at each time step.

A significant difference between the 1/2 Circumcenter Algorithm and the (standard) Circumcenter Algorithm is that, even under complete agent connectivity, convergence does not occur in a single step. This is because $\mathrm{CC}(\{p_i, \frac{p_i+p_j}{2} \,|\, j \in \mathcal{N}_i(\mathcal{G})\})$ does not coincide in general for neighbors.

2.2.1 Modified circumcenter algorithms

Assume now that each agent i is only able to detect a perturbed position, $\overline{p}_j^i \in D(p_j, \sigma)$, of neighbor j. In other words, p_j is the true position of agent j that agent i measures as $\overline{p}_j^i$ such that $\|p_j - \overline{p}_j^i\| \leq \sigma$, for $0 \leq \sigma < r$. In what follows, we assume a centered detection probability over the disk $D(p_j, \sigma)$; that is, $E[\overline{p}_j^i] = p_j$, $\forall\, j \in \mathcal{N}_i(\mathcal{G})$. In particular, this is satisfied by the uniform probability distribution over $D(p_j, \sigma)$ that we consider here.

Due to errors in measurement, an agent that implements a circumcenter algorithm may still lose connectivity with neighbors. If agents have access to a common upper bound of their committed errors, σ, there are two possible corrections they can implement. The first consist of restricting the constraint set where they are allowed to move (variant 1). The second one consists of filtering neighbors' positions (variant 2). For the sake of briefness, we present only the formal description of the Modified 1/2 Circumcenter Algorithm, variants 1 and 2, being the other cases analogous. The algorithms allow each agent to compute $u_i(t)$, $t \in \mathbb{N}$, so that $p_i(t+1) = p_i(t) + u_i(t)$, $i \in \{1, \ldots, n\}$.

Name: Modified 1/2 Circumcenter Algorithm, variant 1

Goal: All agents practically rendezvous

Assumes: (i) $r \in \mathbb{R}_{>0}$ is the sensing radius
(ii) $\sigma < r$ is an upper bound of the sensing errors
(iii) Agents are initially connected by $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$

For $i \in \{1, \ldots, n\}$, agent i executes at $t \in \mathbb{N} \cup \{0\}$:

1: acquire $\{\overline{p}^i_{j_1}, \ldots, \overline{p}^i_{j_{n_i}}\}$,
   such that $\overline{p}^i_j \in D(p_j, \sigma)$, $j \in \mathcal{N}_i(\mathcal{G})$
2: compute
$$\overline{\mathcal{M}}_i := \left\{ \frac{p_i + \overline{p}^i_{j_1}}{2}, \ldots, \frac{p_i + \overline{p}^i_{j_{n_i}}}{2} \right\} \cup \{p_i\}$$
3: compute $\overline{Q}_i := \bigcap_{q \in \overline{\mathcal{M}}_i} D(q, \frac{r-\sigma}{2})$
4: **if** $\overline{Q}_i = \emptyset$ **then**
5: set $u_i := 0$; i.e., stay at p_i
6: **else**
7: set $u_i := \text{CC}(\overline{\mathcal{M}}_i) - p_i$, i.e., move from p_i to $\text{CC}(\overline{\mathcal{M}}_i)$.
8: **end if**

The Modified (standard) Circumcenter Algorithm, variant 1, is similar to the above one with the following substitutions. Step 2 becomes $\overline{\mathcal{M}}_i = \left\{\overline{p}^i_{j_1}, \ldots, \overline{p}^i_{j_{n_i}}\right\} \cup \{p_i\}$, with $j_\ell \in \mathcal{N}_i(\mathcal{G})$, Step 3 becomes $\overline{Q}_i = \bigcap_{q \in \mathcal{M}_i} D(\frac{q+p_i}{2}, \frac{r-\sigma}{2})$, and Step 7 is substituted by (i) the computation of $\lambda^*_i = \lambda(p_i, \text{CC}(\overline{\mathcal{M}}_i), \overline{Q}_i)$; see (1), and (ii) $u_i = \lambda^*_i(\text{CC}(\overline{\mathcal{M}}_i) - p_i)$. Observe that, although the set $\overline{\mathcal{M}}_i$ is different for this algorithm, the motion constraint set $\overline{Q}_i$ is the same.

An alternative to this algorithm is a Modified 1/2 Circumcenter Algorithm, variant 2, that filters the values $\overline{p}^i_j$ as described in the next table. Observe that with variant 2 of the algorithm it is always true that $p_i \in \overline{Q}_i \neq \emptyset$. Again, the Modified (standard) Circumcenter Algorithm, variant 2, is similar to the one in the table with the following substitutions. Step 5 becomes $\overline{\mathcal{M}}_i = \left\{\overline{p}^i_{j_1}, \ldots, \overline{p}^i_{j_{n_i}}\right\} \cup \{p_i\}$, with $j_\ell \in \mathcal{N}_i(\mathcal{G})$, and Step 6 is substituted by (i) compute $\overline{Q}_i = \cap_{q \in \overline{\mathcal{M}}_i} D\left(\frac{q+p_i}{2}, \frac{r}{2}\right)$, (ii) compute $\lambda^*_i = \lambda(p_i, \text{CC}(\overline{\mathcal{M}}_i), \overline{Q}_i)$; see (1), and (iii) apply $u_i = \lambda^*_i(\text{CC}(\overline{\mathcal{M}}_i) - p_i)$. Observe that $p_i \in \overline{Q}_i \neq \emptyset$ always in this case.

Both variants of the circumcenter algorithm, guarantee agent connectivity as shown in the next lemma.

Name: Modified 1/2 Circumcenter Algorithm, variant 2

Goal: All agents practically rendezvous

Assumes: (i) $r \in \mathbb{R}_{>0}$ is the sensing radius
(ii) $\sigma < r$ is an upper bound of the sensing errors
(iii) Agents are initially connected by $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$

For $i \in \{1, \ldots, n\}$, agent i executes at $t \in \mathbb{N} \cup \{0\}$:

1: acquire $\{\overline{p}^i_{j_1}, \ldots, \overline{p}^i_{j_{n_i}}\}$,
   such that $\overline{p}^i_j \in D(p_j, \sigma)$, $j \in \mathcal{N}_i(\mathcal{G})$
2: **for** $\|p_i - \overline{p}^i_j\| > r$, $j \in \mathcal{N}_i(\mathcal{G})$ **do**
3: $\overline{p}^i_j := [p_i, \overline{p}^i_j] \cap D(p_i, r)$
4: **end for**
5: compute
$$\overline{\mathcal{M}}_i := \left\{ \frac{p_i + \overline{p}^i_{j_1}}{2}, \ldots, \frac{p_i + \overline{p}^i_{j_{n_i}}}{2} \right\} \cup \{p_i\}$$
6: set $u_i := \text{CC}(\overline{\mathcal{M}}_i) - p_i$, i.e., move from p_i to $\text{CC}(\overline{\mathcal{M}}_i)$.

Lemma 2 (Connectivity Maintenance) *Let $p_1(t), \ldots, p_n(t) \in \mathbb{R}^d$ be the positions of n agents and suppose that $\|p_i(t) - p_j(t)\| \leq r$ for some $i, j \in \{1, \ldots, n\}$. Let $\sigma < r$ be an upper bound of the errors committed by the agents to measure neighbors' positions. Then, after one execution of any of the circumcenter algorithms proposed above, we have that $\|p_i(t+1) - p_j(t+1)\| \leq r$.*

Proof. We include here the proof for variant 1 of the algorithms, since for variant 2 the proof is similar to the standard case. We have that $p_i(t+1) \in \overline{Q}_i(t) \cup \{p_i(t)\}$ and $p_j(t+1) \in \overline{Q}_j(t) \cup \{p_j(t)\}$. Since it might be possible that $\overline{Q}_k(t) = \emptyset$, for $k \in \{i, j\}$, we distinguish three cases:

(i) If $p_i(t+1) = p_i(t)$ and $p_j(t+1) = p_j(t)$, then it is immediate that $\|p_i(t+1) - p_j(t+1)\| \leq r$.

(ii) If $p_i(t+1) = p_i(t)$ and $p_j(t+1) \neq p_j(t)$, then,

$$\begin{aligned}
&\|p_i(t+1) - p_j(t+1)\| \\
&= \left\| p_i(t+1) - \frac{\overline{p}_i^j(t) + p_j(t)}{2} + \dots \right. \\
&\qquad \left. \dots + \frac{\overline{p}_i^j(t) + p_j(t)}{2} - p_j(t+1) \right\| \\
&\leq \left\| p_i(t+1) - \frac{\overline{p}_i^j(t) + p_j(t)}{2} \right\| \\
&\qquad + \left\| \frac{\overline{p}_i^j(t) + p_j(t)}{2} - p_j(t+1) \right\| \\
&\leq \left\| \frac{p_i(t) - \overline{p}_i^j(t)}{2} \right\| + \left\| \frac{p_i(t) - p_j(t)}{2} \right\| + \frac{r-\sigma}{2} \\
&\leq \frac{\sigma + r + r - \sigma}{2} = r\,.
\end{aligned}$$

(iii) When $p_i(t+1) \neq p_i(t)$ and $p_j(t+1) \neq p_j(t)$,

$$\begin{aligned}
&\left\| p_i(t+1) - p_j(t+1) \right\| = \\
&\left\| p_i(t+1) - \frac{p_i(t) + \overline{p}_j^i(t)}{2} + \frac{p_i(t) + \overline{p}_j^i(t)}{2} \dots \right. \\
&\qquad \left. - \frac{\overline{p}_i^j(t) + p_j(t)}{2} + \frac{\overline{p}_i^j(t) + p_j(t)}{2} - p_j(t+1) \right\| \\
&\leq \left\| p_i(t+1) - \frac{p_i(t) + \overline{p}_j^i(t)}{2} \right\| \dots \\
&\qquad + \left\| \frac{p_i(t) + \overline{p}_j^i(t)}{2} - \frac{\overline{p}_i^j(t) + p_j(t)}{2} \right\| \\
&\qquad + \left\| \frac{\overline{p}_i^j(t) + p_j(t)}{2} - p_j(t+1) \right\| \\
&\leq \frac{r-\sigma}{2} + \left\| \frac{p_i(t) - \overline{p}_i^j(t)}{2} \right\| + \dots \\
&\qquad \left\| \frac{p_j(t) - \overline{p}_j^i(t)}{2} \right\| + \frac{r-\sigma}{2} \leq r\,.
\end{aligned}$$

■

3 Deterministic analysis of the modified circumcenter algorithms

Here we present a deterministic analysis of the convergence of the modified circumcenter algorithms in 1D when using $\mathcal{G}_{\rm disk}(r)$ for some $r \in \mathbb{R}_{>0}$. We will employ the following shorthand notation. Given $\overline{\mathcal{M}}_i = \{p_i, \overline{p}_j^i \mid j \in \mathcal{N}_i(\mathcal{G}_{\rm disk}(r))\}$ and $\mathcal{M}_i = \{p_i, p_j \mid j \in \mathcal{N}_i(\mathcal{G}_{\rm disk}(r))\}$, we denote by $\overline{p}_M^i = \max \overline{\mathcal{M}}_i$ (resp. $p_M^i = \max \mathcal{M}_i$), and $\overline{p}_m^i = \min \overline{\mathcal{M}}_i$ (resp. $p_m^i = \min \mathcal{M}_i$). In this way, $\mathrm{CC}(\overline{\mathcal{M}}_i) = \frac{1}{2}(\overline{p}_M^i + \overline{p}_m^i)$, and $\mathrm{CC}(\mathcal{M}_i) = \frac{1}{2}(p_M^i + p_m^i)$, $i \in \{1, \dots, n\}$. Before stating the main result of the section, we present some properties of the constraint set and Modified (standard) Circumcenter Algorithm, variant 1, in 1D that are employed extensively in the proof of Theorem 5.

Lemma 3 *The constraint set $\overline{Q}_i$ for variant 1 of the modified circumcenter algorithms satisfies:*

(i) $p_i \in \overline{Q}_i$ if and only if $\|p_i - \overline{p}_j^i\| \leq r - \sigma$ for all $j \in \mathcal{N}_i(\mathcal{G}_{disk}(r))$.

(ii) In 1D, we have that $\overline{Q}_i$ is equal to

$$\left[\frac{1}{2}(p_i + \overline{p}_M^i) - \frac{1}{2}(r-\sigma), \frac{1}{2}(p_i + \overline{p}_m^i) + \frac{1}{2}(r-\sigma)\right]$$

and $\overline{Q}_i \neq \emptyset$ if and only if $\|\overline{p}_M^i - \overline{p}_m^i\| \leq 2(r-\sigma)$.

Proof. The proof is straightforward. ■

Lemma 4 *Consider the Modified (standard) Circumcenter Algorithm, variant 1, for n agents initially placed at $p_1(0), \dots, p_n(0) \in \mathbb{R}$. Assume that $q_1 < \mathrm{CC}(\overline{\mathcal{M}}_i(t)) < q_2$ for agent $i \in \{1, \dots, n\}$ and some $q_1, q_2 \in \mathbb{R}$ at some $t \in \mathbb{N} \cup \{0\}$. Then, after one step execution of the algorithm:*

(i) If $\overline{Q}_i(t) \neq \emptyset$ and

$$\frac{\overline{p}_m^i(t) + p_i(t)}{2} + \frac{r-\sigma}{2} - q_1 > 0\,, \qquad (2)$$

then $p_i(t+1) > q_1$.

(ii) If $\overline{Q}_i(t) \neq \emptyset$ and

$$\frac{\overline{p}_M^i(t) + p_i(t)}{2} - \frac{r-\sigma}{2} - q_2 < 0\,.$$

then $p_i(t+1) < q_2$.

(iii) If $\overline{Q}_i(t) = \emptyset$ and $q_1 < p_i(t) < q_2$, then $q_1 < p_i(t+1) < q_2$.

Proof. See that (iii) is immediate because $p_i(t+1) = p_i(t)$ in that case. Let us prove (i). To see that $p_i(t+1) > q_1$ we consider three subcases. *Case (i.a), $p_i(t) > q_1$:* If $p_i(t) > q_1$, $\mathrm{CC}(\overline{\mathcal{M}}_i(t)) > q_1$ and (2) holds, then the $p \in \overline{Q}_i(t) \cup \{p_i(t)\}$ closer to $\mathrm{CC}(\overline{\mathcal{M}}_i(t))$ satisfies $p > q_1$. *Case (i.b), $p_i(t) < q_1$ and $\frac{1}{2}(\overline{p}_M^i(t) + p_i(t)) - \frac{1}{2}(r-\sigma) < q_1$:* In this case, if

(2) holds, then by connectivity of the interval $\overline{Q}_i(t)$ the situation does not differ from case (i.a) regardless of whether $\mathrm{CC}(\overline{\mathcal{M}}_i(t))$ greater or smaller than $\frac{1}{2}(\overline{p}^i_m(t)+p_i(t))+\frac{1}{2}(r-\sigma)$. *Case (i.c), $p_i(t) < q_1$ and $\frac{1}{2}(\overline{p}^i_M(t)+p_i(t))-\frac{1}{2}(r-\sigma) > q_1$:* In particular this implies $\frac{1}{2}(\overline{p}^i_M(t)+p_i(t))-\frac{1}{2}(r-\sigma) > p_i(t)$, which is equivalent to $\overline{p}^i_M(t)-p_i(t) > r-\sigma$. From here,

$$\frac{\overline{p}^i_M(t)+\overline{p}^i_m(t)}{2} - \left(\frac{\overline{p}^i_m(t)+p_i(t)}{2}+\frac{r-\sigma}{2}\right) = \frac{\overline{p}^i_M(t)-p_i(t)-(r-\sigma)}{2} > 0\,.$$

Therefore the $p \in \overline{Q}_i(t) \cup \{p_i(t)\}$ closer to $\mathrm{CC}(\overline{\mathcal{M}}_i(t))$ becomes the upper extreme of $\overline{Q}_i(t)$ which trivially satisfies $p > q_1$. The proof for (ii) is analogous. ■

Theorem 5 *Let $p_1(0),\dots,p_n(0)$ be the initial positions of a robotic network in $\mathbb{R}$. Suppose the agents are initially connected by $\mathcal{G}_{disk}(r)$ for some $r \in \mathbb{R}_{>0}$. Let $\sigma \in \mathbb{R}_{>0}$, $\sigma < r$, be the sensing error radius and $\{P(t) = (p_1(t),\dots,p_n(t))\}_{t\in\mathbb{N}\cup\{0\}}$ a sequence of positions obtained by applying the Modified (standard) Circumcenter Algorithm, variant 1, with $\mathcal{G}_{disk}(r)$. Then, if $r > 7\sigma$, we have $P(t) \to \mathcal{S}_D$, as $t \to \infty$, where*

$$\mathcal{S}_D = \{P \in \mathbb{R}^n \mid \mathrm{diam}(P) \leq 2\sigma\}\,.$$

Proof. We will prove that, if at time $t \in \mathbb{N}$, the agents are not in a ball of diameter 2σ, then $\mathrm{diam}(p_1(t+\ell),\dots,p_n(t+\ell)) < \mathrm{diam}(p_1(t),\dots,p_n(t))$ for either $\ell = 1$ or $\ell = 2$. Without lose of generality suppose that $p_1(t)$ and $p_n(t)$ satisfy $p_1(t) \leq p_i(t) \leq p_n(t)$, for all $i \in \{1,\dots,n\}$. Then the result holds if $p_1(t) < p_i(t+\ell) < p_n(t)$ for all $i \in \{1,\dots,n\}$ and either $\ell = 1$ or $\ell = 2$. We will prove the first inequality holds, being the proof for other one analogous.

If $\mathrm{diam}(p_1(t),\dots,p_n(t)) > 2\sigma$, we can distinguish:

(a) $\exists\, j \in \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$ s. t. $\|p_1(t)-p_j(t)\| > 2\sigma$.
(b) $\forall k \in \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$ we have that $\|p_1(t)-p_k(t)\| \leq 2\sigma$ but $\exists\, j \in \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$ such that $\|p^j_M(t)-p_j(t)\| \leq r$ and $\|p^j_M(t)-p_1(t)\| > r$.

Suppose (a) is true. We will see that Lemma 4 (i) or (iii) holds with $q_1 = p_1(t)$ for all $i \in \{1,\dots,n\}$. Thus, we can say $p_i(t+1) > p_1(t)$, $i \in \{1,\dots,n\}$.

First, $\mathrm{CC}(\overline{\mathcal{M}}_i(t)) > p_1(t)$ for all $i \in \{1,\dots,n\}$:

(a.a) If $i \in \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$, then

$$\mathrm{CC}(\overline{\mathcal{M}}_i(t)) - p_1(t) = \frac{\overline{p}^i_M(t)+\overline{p}^i_m(t)}{2} - p_1(t) \dots$$

$$\dots \geq \frac{p_j(t)-p_1(t)-2\sigma}{2} > 0, \tag{3}$$

since $\overline{p}^i_M(t) \geq p^i_M(t)-\sigma \geq p^1_M(t)-\sigma \geq p_j(t)-\sigma$ and $\overline{p}^i_m(t) \geq p_1(t)-\sigma$.

(a.b) If $i \notin \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r))$, then $p_i(t) > p_j(t)$ and $\overline{p}^i_M(t) > p_j(t)$. Thus, similarly to (a), we have that $\mathrm{CC}(\overline{\mathcal{M}}_i(t)) > p_1(t)$.

Second, (2) holds with $q_1 = p_1(t)$, $i \in \{1,\dots,n\}$:

$$\begin{aligned}\frac{\overline{p}^i_m(t)+p_i(t)}{2} + \frac{r-\sigma}{2} - p_1(t) &\geq \\ \frac{p_1(t)-p_1(t)-\sigma+p_i(t)-p_1(t)+r-\sigma}{2} & \\ \geq \frac{r-2\sigma}{2} > 0\,.\end{aligned}$$

Thus, for those $i \in \{1,\dots,n\}$ such that $p_i(t) \neq p_1(t)$, Lemma 4 (i) or (ii) hold and then $p_i(t+1) > p_1(t)$. To see that $p_i(t+1) > p_1(t)$, for those agents with $p_i(t) = p_1(t)$, we only need to guarantee that $\overline{Q}_i(t) \neq \emptyset$. By Lemma 3, $\overline{Q}_i(t) \neq \emptyset$ if and only if $\overline{p}^i_M(t) - \overline{p}^i_m(t) \leq 2(r-\sigma)$. Since $\overline{p}^i_m(t) \geq p_i(t)-\sigma$, $\overline{p}^i_M(t) \leq p^i_M(t)+\sigma$, then a sufficient condition is given by $\overline{p}^i_M(t)-\overline{p}^i_m(t) \leq \overline{p}^i_M(t)-p_i(t)+\sigma \leq r+2\sigma \leq 2(r-\sigma)$, which holds when $r \geq 4\sigma$. In all, we have seen that under (i), $p_i(t+1) > p_1(t)$ for all $i \in \{1,\dots,n\}$.

Suppose that (b) is true. Let $s \in \{1,\dots,n\}$ such that $p_s(t+1) \leq p_i(t+1)$ for all $i \in \{1,\dots,n\}$. We distinguish two cases:

(b.a) $s \notin \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$,
(b.b) $s \in \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$.

Under (b.a) $s \notin \mathcal{N}_1(\mathcal{G}_{\mathrm{disk}}(r)(t))$, then $p_1(t) < r + p_1(t) < p_s(t)$, $\mathrm{CC}(\overline{\mathcal{M}}_s(t)) = \frac{1}{2}(\overline{p}^s_m(t)+\overline{p}^s_M(t)) \geq \frac{1}{2}(p_1(t)-\sigma+p_s(t)) > p_1(t)+\frac{1}{2}(r-\sigma) > p_1(t)$ and, similarly, $\frac{1}{2}(\overline{p}^s_m(t)+p_s(t))+\frac{1}{2}(r-\sigma) > p_1(t)$. Thus Lemma 4 implies $p_1(t) < p_s(t+1) \leq p_i(t+1)$, $i \in \{1,\dots,n\}$, and thus the result holds for $\ell = 1$.

Under (b.b), we will apply Lemma 4 for $p_i(t+2)$ and with respect to $q_1 = p_1(t)$. To do so, we need to find several lower bounds. First, let $j \in \mathcal{N}_1(\mathcal{G}_{\text{disk}}(r)(t))$ be such that $p_j(t) = p_M^1(t)$; i.e., $p_j(t)$ is the furthest away position from $p_1(t)$. By assumption (b), $p_M^j(t)$ is such that $\|p_M^j(t) - p_j(t)\| \leq r$ and $\|p_M^j(t) - p_1(t)\| > r$. Let us determine a lower bound for $p_j(t+1) - p_1(t)$.

Using similar bounds as in (3), $\text{CC}(\overline{\mathcal{M}}_j(t)) \geq \frac{1}{2}(p_M^j(t) + p_1(t)) - \sigma$. On the other hand $p_j(t+1) \in \overline{Q}_j(t) \cup \{p_j(t)\}$. We can guarantee that

$$\frac{\overline{p}_M^j(t) + p_j(t)}{2} - \frac{r-\sigma}{2} \leq \frac{p_M^j(t) + p_1(t)}{2} - \sigma \,,$$

if the following inequality holds:

$$\frac{p_M^j(t) + p_j(t) + \sigma}{2} - \frac{r-\sigma}{2} \leq \frac{p_M^j(t) + p_1(t)}{2} - \sigma \,,$$

which simplifies to $\frac{1}{2}(r - 4\sigma) \geq \frac{1}{2}(p_j(t) - p_1(t))$. Since $p_j(t) - p_1(t) \leq 2\sigma$, we can guarantee the previous inequality by taking $r - 4\sigma \geq 2\sigma$, or $r \geq 6\sigma$. We also have that $\overline{Q}_j(t) \neq \emptyset$ if $r \geq 6\sigma$. In fact, by Lemma 4, $\overline{Q}_j(t) \neq \emptyset$ if and only if $\overline{p}_M^j(t) - \overline{p}_m^j(t) \leq 2(r - \sigma)$. Since $\overline{p}_m^j(t) \geq p_1(t) - \sigma$ and $\overline{p}_M^j(t) \leq p_M^j(t) + \sigma$, a sufficient condition is given by $p_M^j(t) - p_1(t) + 2\sigma \leq 2(r - \sigma)$. Further, since $p_M^j(t) - p_j(t) \leq r$ and $p_j(t) - p_1(t) \leq 2\sigma$, it is sufficient $r + 4\sigma \leq 2r - 2\sigma$, or $r \geq 6\sigma$.

Assuming then $r \geq 6\sigma$, we distinguish two cases:

- Suppose first that the upper extreme of $\overline{Q}_j(t)$, is such that $\frac{1}{2}(\overline{p}_m^j(t) + p_j(t)) + \frac{1}{2}(r - \sigma) \leq \frac{1}{2}(p_M^j(t) + p_1(t)) - \sigma$. Observe that, since $p_M^j(t) \notin \mathcal{N}_1(\mathcal{G}_{\text{disk}}(r)(t))$ and $p_j(t) - p_1(t) \leq 2\sigma$, it must be $p_M^j(t) - p_j(t) > 4\sigma$ and thus $\frac{1}{2}(p_M^j(t) + p_1(t)) - \sigma - p_j(t) > 0$. Because $\text{CC}(\overline{\mathcal{M}}_j(t)) \geq \frac{1}{2}(p_M^j(t) + p_1(t)) - \sigma$, then $p_j(t+1) \in \max\{\frac{1}{2}(\overline{p}_m^j(t) + p_j(t)) + \frac{r-\sigma}{2}, p_j(t)\}$ and

$$\begin{aligned} p_j(t+1) - p_1(t) &\geq \frac{\overline{p}_m^j(t) + p_j(t)}{2} + \frac{r-\sigma}{2} - p_1(t) \\ &\geq \frac{p_1(t) - \sigma - p_1(t) + p_j(t) - p_1(t)}{2} + \frac{r-\sigma}{2} \\ &\geq \frac{p_j(t) - p_1(t)}{2} + \frac{r - 2\sigma}{2} > \frac{r-2\sigma}{2} \,. \end{aligned}$$

- When $\frac{1}{2}(\overline{p}_m^j(t) + p_j(t)) + \frac{1}{2}(r - \sigma) \geq \frac{1}{2}(p_M^j(t) + p_1(t)) - \sigma$, then

$$p_j(t+1) - p_1(t) \geq \frac{p_M^j(t) - p_1(t)}{2} - \sigma > \frac{r - 2\sigma}{2} \,.$$

In both cases, we have that $p_j(t+1) - p_1(t) > (r - 2\sigma)/2$ when $r > 6\sigma$.

Let us now compute a lower bound for $p_i(t+1) - p_i(t)$, $i \in \{1, \dots, n\}$, using Lemma 4. We have that $p_i(t) > p_1(t) - \sigma/2$, $\text{CC}(\overline{\mathcal{M}}_i(t)) = \frac{1}{2}(\overline{p}_M^i(t) + \overline{p}_m^i(t)) \geq \frac{1}{2}(p_i(t) + p_1(t) - \sigma) > p_1(t) - \sigma/2$, and, similarly $\frac{1}{2}(\overline{p}_m^i(t) + p_i(t)) + \frac{1}{2}(r - \sigma) > p_1(t) - \sigma/2$. By Lemma 4, this implies $p_i(t+1) \geq p_1(t) - \sigma/2$, for all $i \in \{1, \dots, n\}$.

Now, with the help of the obtained bounds, we can lower bound $\text{CC}(\overline{\mathcal{M}}_i(t+1)) - p_1(t)$, $i \in \{1, \dots, n\}$, as follows. If $i \in \mathcal{N}_j(\mathcal{G}_{\text{disk}}(r)(t+1))$, then $p_M^i(t+1) \geq p_M^1(t+1) \geq p_j(t+1)$ and

$$\begin{aligned} \text{CC}(\overline{\mathcal{M}}_i(t+1)) &= \frac{\overline{p}_M^i(t+1) + \overline{p}_m^i(t+1)}{2} \\ &\geq \frac{1}{2}(p_j(t+1) - \sigma + p_s(t+1) - \sigma) \,. \end{aligned} \tag{4}$$

On the other hand, for those i such that $\|p_i(t+1) - p_j(t+1)\| > r$, we have that $p_i(t+1) > p_j(t+1)$. Otherwise, because the algorithm preserves agent connectivity, and $s \in \mathcal{N}_j(\mathcal{G}_{\text{disk}}(r))$, $0 < p_j(t+1) - p_i(t+1) \leq p_j(t+1) - p_s(t+1) \leq r$, which is a contradiction. Then an inequality like (4) holds for every $i \in \{1, \dots, n\}$. In this way,

$$\begin{aligned} &\text{CC}(\overline{\mathcal{M}}_i(t+1)) - p_1(t) \\ &\geq \frac{1}{2}(p_j(t+1) - p_1(t)) - \frac{\sigma}{2} + \frac{1}{2}(p_s(t+1) - p_1(t)) - \frac{\sigma}{2} \\ &> \frac{1}{2}\left(\frac{r-2\sigma}{2}\right) - \sigma - \frac{\sigma}{4} = \frac{r - 7\sigma}{2} \,. \end{aligned}$$

That is, $\text{CC}(\overline{\mathcal{M}}_i(t+1)) > p_1(t)$ when $r > 7\sigma$. On the other hand, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} &\frac{\overline{p}_m^i(t+1) + p_i(t+1)}{2} + \frac{r-\sigma}{2} - p_1(t) \\ &= \frac{\overline{p}_m^i(t+1) - p_1(t) + p_i(t+1) - p_1(t)}{2} + \frac{r-\sigma}{2} \\ &\geq \frac{-\frac{\sigma}{2} - \sigma - \frac{\sigma}{2}}{2} + \frac{r-\sigma}{2} \geq \frac{r - 3\sigma}{2} > 0 \,. \end{aligned}$$

In order to apply Lemma 4, it remains to be proven that for those $p_i(t+1) \leq p_1(t)$ we have $\overline{Q}_i(t+1) \neq \emptyset$.

Since $p_s(t+1) \geq p_1(t) - \sigma/2$, then $\overline{p}^i_m(t+1) \geq p_1(t) - 3\sigma/2$. In this way, $\overline{p}^i_M(t+1) - \overline{p}^i_m(t+1) \leq \overline{p}^i_M(t+1) - p_1(t) + 3\sigma/2 \leq \overline{p}^i_M(t+1) - p_i(t+1) + 3\sigma/2 \leq r + \sigma + 3\sigma/2$. A sufficient condition for $\overline{Q}_i(t+1) \neq \emptyset$ is then given by $r + 5\sigma/2 \leq 2(r - \sigma)$, or $r \geq 9\sigma/2$.

Thus, by Lemma 4, we have that $p_i(t+2) > p_1(t)$ for all $i \in \{1, \dots, n\}$. The discussion on the right end $p_n(t)$ is analogous to the one presented for $p_1(t)$.

Using (i) and (ii) we can find a subsequence of $\mathrm{diam}(p_1(t), \dots, p_n(t))$, $t \in \mathbb{N} \cup \{0\}$, that decreases as long as $\mathrm{diam}(p_1(t), \dots, p_n(t)) > 2\sigma$. Using a LaSalle type of argument with $\mathrm{diam}(p_1, \dots, p_n)$ as a Lyapunov function, we can conclude that $P(t) \to \mathcal{S}_D$, as $t \to +\infty$. ■

Remark 6 This result holds independently of the number of agents in the network, which in particular does not affect the diameter 2σ of the practical stability ball. As we show in simulations later, the ball does wander in space by the effect of noise. The theorem gives only sufficient conditions for decreasing the diameter strictly after two time steps. Simulations also show convergence for smaller ratios r/σ.•

Remark 7 Observe that the proof of convergence makes explicit use of the fact that the connectivity graph is $\mathcal{G}_{\mathrm{disk}}(r)$ in item (b). As we show in simulations later, convergence to a practical stability ball of a similar radius can be observed for other graphs as well. The proof of Theorem 5 holds to prove stability of variant 2 of the algorithm, with the difference that we do not need to make sure that $\overline{Q}_i \neq \emptyset$ since this property is always true. Finally, the proof can also be simplified for the 1/2 Circumcenter Algorithm, variant 1 and 2. In this case, we do not need to use Lemma 4, since the motion of agents is to $\mathrm{CC}(\overline{\mathcal{M}}_i) = \frac{\overline{p}^i_M + 2p_i + \overline{p}^i_m}{4}$ whenever the constraint set $\overline{Q}_i$ is not empty. The analogous proof method leads to a required $r/\sigma > 19$. The higher ratio is due to the slower motion of agents to new circumcenters. •

Remark 8 The deterministic analysis of the algorithm guarantees performance in 1D dimensions. Although restrictive, the analysis can also be used to guarantee performance of the Modified Parallel Circumcenter algorithms, valid in higher dimensions. Recall that a requirement for the implementation of this algorithm is that agents have knowledge of a common reference frame. •

4 Stochastic analysis of the Modified Circumcenter Algorithms

In this section we present a stochastic analysis of the proposed circumcenter algorithms. The main result relies on the next Supermartingale Convergence theorem taken from [15], and some previous lemmas.

Theorem 9 (Supermartingale Convergence Th) *Suppose that X_t, $t \in \mathbb{N} \cup \{0\}$, is a nonnegative random variable such that $E[X_1] < +\infty$. Let $\mathcal{F}_t$ denote the history of process X_t up to time $t \in \mathbb{N} \cup \{0\}$. If*

$$E[X_{t+1} | \mathcal{F}_t] \leq X_t, \qquad w.p.1, \text{ then}$$

X_t tends to a limit X w.p.1. and $\lim_{t \to +\infty} E[X_t] = E[X]$.

Lemma 10 *Let $p_1, \dots, p_n$ be the n vertices of a convex polygon ordered in a counterclockwise manner. Then, there exists a vertex p_i such that the angle of the polygon at this vertex, $\alpha_i = \angle(p_{i-1}, p_i, p_{i+1})$, is bounded by $(n-2)\pi/n < \pi/2$.*

Proof. Divide the convex polygon into $n-2$ non-overlapping triangles with common vertex p_i. Now each angle α_j at a vertex p_j can be obtained as the sum of the angles at p_j of those triangles that have p_j as a vertex. Since the sum of the angles of a triangle is exactly π and we have $n-2$ triangles then $\sum_{i \in \{1,\dots,n\}} \alpha_i = (n-2)\pi$. Now suppose that all the angles are lower bounded strictly by $(n-2)\pi/n$. That would imply that $(n-2)\pi < \sum_{j \in \{1,\dots,n\}} \alpha_j = (n-2)\pi$, which is a contradiction. ■

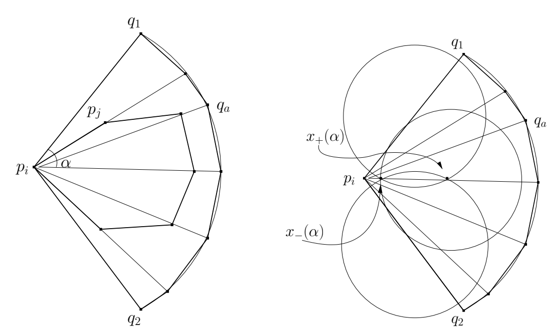

Fig. 1. The figure on the left shows the location of q_1 and q_2. The figure on the right shows the intersection of several disks with radius $\frac{r-\sigma}{2}$ and centers $\frac{p_1+q_a}{2}$.

Lemma 11 *Let $p_1, \dots, p_n$ be the n vertices of a convex polygon ordered in a counterclockwise manner. Let $r > \sigma > 0$ and $k > 0$. Suppose that $\alpha_1 = \angle(q_1, p_1, q_2) \leq 2\alpha = (n-2)\pi/n$, that q_1, q_2 are at* *a distance $r-\sigma+k$ from p_1, and that the sector of $D(p_1, r-\sigma+k)$ limited by the angle $\angle(q_1, p_1, q_2)$ contains all other p_j, $j \in \{1, \dots, n\}$; see Figure 1.*

(i) The following set content holds:

$$I_0 = D\left(p_1, \frac{r-\sigma}{2}\right) \cap D\left(\frac{p_1+q_1}{2}, \frac{r-\sigma}{2}\right) \cap \dots \cap D\left(\frac{p_1+q_2}{2}, \frac{r-\sigma}{2}\right) \subseteq \bigcap_{i=1}^{n} D\left(\frac{p_1+p_j}{2}, \frac{r-\sigma}{2}\right)$$

(ii) If $k \le \min\left\{(r-\sigma)\frac{1-\sin\alpha}{\sin\alpha}, (r-\sigma)\frac{1-\cos\alpha}{\cos\alpha}\right\}$, *then* $I_0 \neq \emptyset$.

Proof. Let us prove (i) for any $d = \|q_i - p_1\|$, $i \in \{1, 2\}$, and disks of radius s with $2s < d$; which holds for $2\frac{r-\sigma}{2} < r-\sigma+k$. Since p_j, $j \in \{1, \dots, n\}$, are in the circular sector of $D(p_1, d)$ limited by $\angle(q_1, p_1, q_2)$, we can find a finite number of points $q_a \in \mathrm{arc}_{\mathbb{S}(p_1,d)}(q_2, q_1)$, $a \in \mathcal{A}$, such that $\mathrm{co}\{p_1, \dots, p_n\} \subseteq \mathrm{co}\{p_1, q_a \,|\, a \in \mathcal{A}\}$. This implies $\frac{p_1+p_j}{2} \in \mathrm{co}\{\frac{p_1+q_a}{2} | a \in \mathcal{A}\}$, $j \in \{1, \dots, n\}$, and we can write $\frac{p_1+p_j}{2} = \sum_{a\in\mathcal{A}} \lambda_a^j \left(\frac{p_1+q_a}{2}\right)$, for some $\lambda_a^j \ge 0$, such that $\sum_{a\in\mathcal{A}} \lambda_a^j = 1$. Then, for every $p \in \bigcap_{a\in\mathcal{A}} D(\frac{p_1+q_a}{2}, s)$ we have

$$\left\|p - \frac{p_1+p_j}{2}\right\| = \left\|\sum_{a\in\mathcal{A}} \lambda_a^j (p - \frac{p_1+q_a}{2})\right\| \le \sum_{a\in\mathcal{A}} \lambda_a^j \left\|p - \frac{p_1+q_a}{2}\right\| \le \sum_{a\in\mathcal{A}} \lambda_a^j s = s\,.$$

Thus, $D(p_1, s) \cap \bigcap_{a\in\mathcal{A}} D(\frac{p_1+q_a}{2}, s) \subseteq \bigcap_{j=1}^{n} D(\frac{p_1+p_j}{2}, s)$. Now let us see that $I_0 \subseteq D(p_1, s) \cap \bigcap_{a\in\mathcal{A}} D(\frac{p_1+q_a}{2}, s)$.

Consider the intersection of a disk $D(\frac{p_1+q_a}{2}, s)$ with the bisector of the angle $\angle(q_1, p_1, q_2)$. Without loss of generality we can assign coordinates $p_1 = (0, 0)$ and $q_a = (d\cos\beta, d\sin\beta)$ with $0 \le \beta \le 2\alpha$. It is easy to see that the intersection points $x_-(\beta)$ and $x_+(\beta)$ have coordinates:

$$x_+(\beta) = \left(\frac{d\cos\beta}{2} + \frac{1}{2}\sqrt{4s^2 - d^2\sin^2\beta}, 0\right)\,,$$
$$x_-(\beta) = \left(\frac{d\cos\beta}{2} - \frac{1}{2}\sqrt{4s^2 - d^2\sin^2\beta}, 0\right)\,,$$

which are in $\mathbb{R}^2$ when $2s > d\sin\beta$. The first components of these vectors are increasing and decreasing functions of $\beta \in [0, \pi/2]$ when $d/2 > s$:

$$x_+^{1'}(\beta) = -\frac{d\sin\beta}{2} - \frac{d^2\sin\beta\cos\beta}{2\sqrt{4s^2 - d^2\sin^2\beta}} < 0\,,$$
$$x_-^{1'}(\beta) = -\frac{d\sin\beta}{2} + \frac{d^2\sin\beta\cos\beta}{2\sqrt{4s^2 - d^2\sin^2\beta}} > 0\,,$$

In case $d = r-\sigma+k$, $s = \frac{r-\sigma}{2}$ and $k < \frac{(r-\sigma)(1-\sin\alpha)}{\sin\alpha}$, these inequalities are always true. In particular, $x_+^1(\alpha) \le x_+^1(\beta)$ and $x_-^1(\alpha) \ge x_-^1(\beta)$ for all $0 \le \beta \le 2\alpha$. Since we are intersecting disks of the same curvature, this means that the region of $D(\frac{p_1+q_1}{2}, s)$ enclosed by moving counterclockwise from $x_+(\alpha)$ to $x_-(\alpha)$ along $[x_+(\alpha), x_-(\alpha)]$ and $\mathrm{arc}_{\mathbb{S}(\frac{p_1+q_1}{2},s)}(x_-(\alpha), x_+(\alpha))$; see Figure 1, is contained in the disks $D(\frac{p_1+q_a}{2}, s)$ for all $a \in \mathcal{A}$. The same can be said of the region of $D(\frac{p_1+q_2}{2}, s)$ enclosed by moving clockwise from $x_+(\alpha)$ to $x_-(\alpha)$ along $[x_-(\alpha), x_+(\alpha)]$ and $\mathrm{arc}_{\mathbb{S}(\frac{p_1+q_2}{2},s)}(x_+(\alpha), x_-(\alpha))$. Since $D(\frac{p_1+q_1}{2}, s) \cap D(\frac{p_1+q_2}{2}, s)$ is exactly the union of these two regions, we have that $I_0 \subseteq D(p_1, s) \cap \bigcap_{a\in\mathcal{A}} D(\frac{p_1+q_a}{2}, s)$.

Now let us prove (ii) for $d = r-\sigma+k$ and $2s = r-\sigma$. According to Lemma 1, $I_0 \neq \emptyset$ if and only if the circumcenter $\mathrm{CC}(p_1, \frac{p_1+q_1}{2}, \frac{p_1+q_2}{2})$, is in I_0. Thus, in order for $I_0 \neq \emptyset$, it is enough to guarantee that the circumradius $\mathrm{CR}(p_1, \frac{p_1+q_1}{2}, \frac{p_1+q_2}{2})$, is less or equal to s. In that case $\mathrm{CC}(p_1, \frac{p_1+q_1}{2}, \frac{p_1+q_2}{2})$ is at a distance from the vertices not greater than s, and therefore $\mathrm{CC}(p_1, \frac{p_1+q_1}{2}, \frac{p_1+q_2}{2}) \in I_0$.

Since the triangle formed by p_1, $\frac{p_1+q_1}{2}$, and $\frac{p_1+q_2}{2}$ is isosceles, see Figure 1, we can find a center at $C = (\frac{d}{2}\cos\alpha, 0)$ and a radius $R = \max\{\frac{d}{2}\sin\alpha, \frac{d}{2}\cos\alpha\}$ such that the triangle formed by p_1, $\frac{p_1+q_1}{2}$, and $\frac{p_1+q_2}{2}$ is contained in the disk $D(C, R)$. This implies that $\mathrm{CR}(p_1, \frac{p_1+q_1}{2}, \frac{p_1+q_2}{2}) \le R$. Imposing $R \le s$, leads to

$$\frac{(r-\sigma+k)}{2}\sin\alpha \le \frac{r-\sigma}{2}\,,$$
$$\frac{(r-\sigma+k)}{2}\cos\alpha \le \frac{r-\sigma}{2}\,.$$

This is guaranteed if

$$k \le \min\left\{(r-\sigma)\frac{1-\sin\alpha}{\sin\alpha}, (r-\sigma)\frac{1-\cos\alpha}{\cos\alpha}\right\}\,.$$

From here the proof of item (ii) follows. ■

Theorem 12 *Let $p_1(0),\dots,p_n(0)$ be the initial positions of a robotic network in $\mathbb{R}^2$. Suppose the agents are initially connected by $\mathcal{G}_{disk}(r)$ for some $r\in\mathbb{R}_{>0}$. Let $\sigma\in\mathbb{R}_{>0}$, $\sigma<r$, be the sensing error radius, and let $\{P_t=(p_1(t),\dots,p_n(t))\}_{t\in\mathbb{N}\cup\{0\}}$ denote a sequence of positions obtained by applying the Modified $1/2$ Circumcenter Algorithm. Let V_t denote the random variable $V_t=\mathrm{diam}(P_t)$, $t\in\mathbb{N}\cup\{0\}$. Then, for any $r>\sigma$ we have that V_t converges to a limit V w.p.1 such that $E[V]=0$.*

Proof. We will apply Theorem 9 to the nonnegative random variable V_t. Clearly, for any fixed number of agents n, we will have that $E[V_1]<+\infty$. To establish an inequality as in Theorem 9, recall the following. The local convex hull of an agent $i\in\{1,\dots,n\}$ and neighbors is defined as

$$\begin{aligned}S_i&=\mathrm{co}\{p_i,p_{j_s}\mid j_s\in\{1,\dots,n_i\}\}\\&=\{\lambda_{j_0}p_i+\lambda_{j_1}p_{j_1}+\dots+\lambda_{j_{n_i}}p_{j_{n_i}}\,,\\&\qquad\mid\lambda_{j_s}\in[0,1],\ s\in\{1,\dots,n_i\},\ \sum_{s=0}^{n_i}\lambda_{j_s}=1\},\end{aligned}$$

where n_i denotes the number of neighbors of agent i. Consider now any measurement distribution of neighbors $\overline{p}_j^i$ such that $E[\overline{p}_j^i]=p_j$. The local convex hull measured by agent i is a random variable given by $\overline{S}_i=\mathrm{co}\{p_i,\overline{p}_{j_s}^i\mid s\in\{1,\dots,n_i\}\}$. Since taking expected values is a linear operation, we have that $E[\overline{S}_i]=S_i$ for all $i\in\{1,\dots,n\}$. In this way, for any of modified circumcenter algorithms we have $E[V_{t+1}\big|\mathcal{F}_t]\le V_t$ for all $t\ge 0$. Theorem 9 applies and we can say that the process V_t will converge to a stationary value V with probability one.

We prove next that, w.p.1 there exists a sequence of times t_ℓ, $\ell\in\mathbb{N}$, such that $E[V_{t_\ell}\big|\mathcal{F}_{t_\ell-1}]<V_{t_\ell-1}$, for all $\ell\in\mathbb{N}$. Without loss of generality suppose $t=0$ and let us find $t>0$ satisfying the strict inequality.

In the variant 2 of the algorithm, we have that $p_i(1)\neq p_i(0)$ and $p_i(1)$ coincides with the circumcenter of $\overline{\mathcal{M}}_i=\left\{p_i,\frac{p_i+\overline{p}_j^i}{2}\mid j\in\mathcal{N}_i\right\}$ (no constraint restricted by $r-\sigma$ is enforced). This implies that $p_i(1)\in\overline{S}_i(0)\setminus\{p_i(0),\overline{p}_j^i(0)\mid j\in\mathcal{N}_i(0)\}$ and therefore $E[p_i(1)]\in S_i(0)\setminus\{p_i(0),p_j(0)\mid j\in\mathcal{N}_i(0)\}$. Thus, $E[V_1\big|\mathcal{F}_0]<V_0$.

For the variant 1 of the algorithm, we can not guarantee in general that $\overline{Q}_i\neq\emptyset$ for all $i\in\{1,\dots,n\}$ and thus agents may remain stationary. Reasoning by contradiction we will prove that there exists at least an agent i, such that $p_i(0)\in\partial(\mathrm{co}\{p_1(0),\dots,p_n(0)\})$, with $p_i(0)$ determining the diameter of $\mathrm{co}\{p_1(0),\dots,p_n(0)\}$, and a time $t>0$ such that $\overline{Q}_i(t)\neq\emptyset$ with probability one. This implies that w.p.1 there exists a time $t>0$ such that $E[V_{t+1}|\mathcal{F}_t]<V_0$.

Assume that $\overline{Q}_j(t)=\emptyset$ for all $t>0$ and $j\in\{1,\dots,n\}$. As a consequence, none of the agents will be able to move. By Lemma 11, there exists $i\in\{1,\dots,n\}$, such that $p_i(0)\in\partial(\mathrm{co}\{p_1(0),\dots,p_n(0)\})$, $p_i(0)$ determines the diameter of $\mathrm{co}\{p_1(0),\dots,p_n(0)\}$, and the angle formed by $p_i(0)$ and the positions of any other two agents $p_k(0)$, $p_j(0)$, is upper bounded by $\angle(p_i(0),p_j(0),p_k(0))\le(m-2)\pi/m\le 2\alpha=(n-2)\pi/n<\pi/2$.

For a uniform distribution of $\overline{p}_j^i$ in a disk of radius σ and centered at p_j, the probability that $\|p_i(t)-\overline{p}_j^i(t)\|\le r-\sigma+k$, for any $k>0$, is a positive one:

$$P(\|p_i(t)-\overline{p}_j^i(t)\|\le r-\sigma+k)=\begin{cases}1, & \|p_i(t)-p_j(t)\|\le r-2\sigma\,,\\ \frac{1}{\pi\sigma^2}\int_D dp>0, & r-2\sigma\le\|p_i(t)-p_j(t)\|\le r.\end{cases}$$

where $D=\{p\in D(p_j,\sigma)\mid\|p_i-p\|\le r-\sigma+k\}$. To see this, observe that $\|p_j(t)-p_i(t)\|\le r$. This implies that at least there is a point of $D(p_j(t),\sigma)$ at distance $r-\sigma$ of $p_i(t)$ and a subset $M\subset D(p_j(t),\sigma)$ with nonzero measure such that $\|p-p_i(t)\|\le r-\sigma+k$ for every $p\in M$. Let us use the notation $P(\|p_i(t)-\overline{p}_j(t)\|\le r-\sigma+k)=a>0$ and let us compute the probability of the event $A=\{\exists t>0\mid\|p_i(t)-\overline{p}_{j_1}(t)\|\le r-\sigma+k\}$ for a fixed $j_1\in\mathcal{N}_i(\mathcal{G}_{\mathrm{disk}}(r))$. In fact, we can write A as the disjoint union of events A_t, $t\in\mathbb{N}\cup\{0\}$:

$$\begin{aligned}A=\cup_{t=0}^{\infty}A_t=\cup_{t=0}^{\infty}\{&\|p_i(s)-\overline{p}_{j_1}(s)\|>r-\sigma+k,\\&\forall s\le t-1,\text{ and }\|p_i(t)-\overline{p}_{j_1}(t)\|\le r-\sigma+k\}\,.\end{aligned}$$

In this way,

$$P(A)=\sum_{t=0}^{\infty}P(A_t)=\sum_{t=0}^{\infty}a(1-a)^t=\frac{a}{1-(1-a)}=1.$$

Reasoning in an inductive manner, we can find w.p.1 an infinite sequence of times t_ℓ, $\ell\in\mathbb{N}$, such that $\|p_i(t_\ell)-p_{j_1}(t_\ell)\|\le r-\sigma+k$, for all $\ell\in\mathbb{N}$. Similarly, we can find w.p.1 an infinite subsequence of times

t_{ℓ_m}, $m \in \mathbb{N}$, such that $\|p_i(t_\ell) - p_{j_1}(t_\ell)\| \leq r - \sigma + k$ and $\|p_i(t_\ell) - p_{j_2}(t_\ell)\| \leq r - \sigma + k$ for another neighbor $j_2 \in \mathcal{N}_i(\mathcal{G}_{\text{disk}}(r))$. Since the number of neighbors of i is finite, we can extend this argument to conclude that w.p.1 there exists a $t > 0$ such that for every $j \in \mathcal{N}_i(\mathcal{G}_{\text{disk}}(r))$ we have that $\|p_i(t) - \overline{p}^i_j(t)\| \leq r - \sigma + k$. How large t is will depend on the number of neighbors of i and how large k is.

Now, by choosing k as in Lemma 11 (ii) we can guarantee that w.p.1 there exist a $t > 0$ such that $\overline{Q}_i(t) \neq \emptyset$. This implies that w.p.1 $E[V_{t+1} \big| \mathcal{F}_t] < V_0$ for some $t > 0$. From here the result follows. ■

Corollary 13 (Extension to switching graphs) *Let $p_1(0), \ldots, p_n(0)$ be the initial positions of a robotic network in $\mathbb{R}^2$. Suppose the agents are connected by a sequence of graphs $\mathcal{G}(t) \subseteq \mathcal{G}_{disk}(r)$, $t \in \mathbb{N} \cup \{0\}$, such that for some $T \in \mathbb{N}$, $\mathcal{G}(Tk)$ is strongly connected, with $k \in \mathbb{N}$. Let $\sigma \in \mathbb{R}_{>0}$, $\sigma < r$, be the sensing error radius, and let $\{P_t = (p_1(t), \ldots, p_n(t))\}_{t \in \mathbb{N} \cup \{0\}}$ denote a sequence of positions obtained by applying the Modied 1/2 Circumcenter Algorithm, variant 1. Let V_t denote the random variable $V_t = \text{diam}(P_t)$, $t \in \mathbb{N} \cup \{0\}$. Then for any $r > \sigma$ we have that V_t converges to a limit w.p.1 such that $E[V] = 0$.*

Proof. The proof of this fact is very similar to that of the previous theorem. First, observe that the argument of the proof can be extended for any fixed strongly connected graph $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$. Now consider that the algorithm is implemented over a sequence of graphs $\mathcal{G}(t)$ that contains a subsequence of strongly connected graphs $\{\mathcal{G}(Tk)\} \subseteq \{\mathcal{G}(t)\}$, for some $T > 0$. To extend the proof, we just need to see that with probability one there exists a subsequence of times $\{Tk_m\} \subset \{Tk\}$ such that $\overline{Q}_i(t) \neq \emptyset$ for some $i \in \{1, \ldots, n\}$ and $p_i(0)$ satisfying similar properties as in the theorem. Take one of the agents $j \in \mathcal{N}_i(\mathcal{G}_{\text{disk}}(r))$, which are neighbors of i for an infinite number of times; i.e. $j \in \mathcal{N}_i(\mathcal{G}(Tk_{m_j}))$, with $m_1 \in \mathbb{N}$. Since the graphs $\mathcal{G}(Tk)$ are strongly connected, there exist at least one of such agents. Now for every instant Tk_{m_1}, consider another neighbor $j_2 \in \mathcal{N}_i(\mathcal{G}(Tk_{m_1}))$ that appears an infinite number of times (if there is none we have finished). That determines another infinite sequence of graphs $\mathcal{G}(Tk_{m_2})$. We repit this process until we have a fixed collection of neighbors $j_1, \ldots, j_s \in \mathcal{G}(Tk_{m_s})$ which are all the neighbors that i has at infinite instants Tk_{m_s}, $s \in \mathbb{N}$. For this set of neighbors we can repit the argument of the proof in the theorem to conclude that, w.p.1, there exists a time $t > 0$ such that $\overline{Q}_i(t) \neq \emptyset$. The time of convergence is further affected by T and the switching of graphs. ■

Remark 14 The extension of the previous proof to the (standard) Circumcenter Algorithm, variant 1, becomes more difficult as it requires checking that the closest point to $E[\text{CC}(\overline{\mathcal{M}_i})]$ does not contain any vertex $p_j(0)$. Despite this, all the simulations in 2D showed multiagent rendezvous to a practical stability ball. On the other hand; see [9], the stochastic analysis of the algorithm in 1D becomes easier as we have that $E[\frac{1}{2}(\overline{p}^i_M + \overline{p}^i_m)] = \frac{1}{2}(p^i_M + p^i_m)$ for all $i \in \{1, \ldots, n\}$. Proving this fact in 2D has been elusive (in several dimensions we have that $\text{CC}(S_i) = \sum_{s=1}^{n_i} \lambda_{j_s}(p_{j_1}, \ldots, p_{j_{n_i}}) p_{j_s}$, which is a nonlinear function of the p_{j_s}.) These difficulties disappear when considering variant 2 of the (standard) Circumcenter Algorithm. In this case the constraint sets are always non-empty and contain always $p_i(0)$ (the analysis would be similar to a noiseless (standard) Circumcenter Algorithm). •

Remark 15 Let $p_1(0), \ldots, p_n(0)$ be the initial positions of a robotic network in $\mathbb{R}^2$. Then, under the Modified (standard) Circumcenter Algorithm, variant 1, and for $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$ agents reach a ball of radius $\frac{r-\sigma}{\sqrt{2}}$. This implies $\|p_i - p_j\| \leq r$ for all $i, j \in \{1, \ldots, n\}$.

To see this suppose that $E[\text{diam}(p_1(t+1), \ldots, p_n(t+1)) | (p_1(t), \ldots, p_n(t))] = 0$. Then

$$\begin{aligned} 0 =& E[\text{diam}(p_1(t+1), \ldots, p_n(t+1)) \,|\, \mathcal{P}(t)] \\ &= \max_{i,j} E[\|p_i(t+1) - p_j(t+1)\| \,|\, \mathcal{P}(t)] \\ &\geq \max_{i,j} \|E[p_i(t+1) \,|\, \mathcal{P}(t)] - E[p_j(t+1) \,|\, \mathcal{P}(t)]\| \,, \end{aligned}$$

and $E[p_i(t+1)|\mathcal{P}(t)] = E[p_j(t+1)|\mathcal{P}(t)] = p$ for all $i, j \in \{1, \ldots, n\}$. This implies that

$$\begin{aligned} p &\in \bigcap_{j=1}^{n} \left(\{p_j(t)\} \cup E[\overline{Q}_j(t) \,|\, \mathcal{P}(t)] \right) \quad (5) \\ &\subseteq \bigcap_{j=1}^{n} (\{p_j(t)\} \cup \bigcap_{i \in \mathcal{N}_j(\mathcal{G})} E[D\left(\frac{p_j(t+1) + \overline{p}^j_i(t+1)}{2}, \frac{r-\sigma}{2}\right)]) \\ &\subseteq \bigcap_{j=1}^{n} (\{p_j(t)\} \cup \bigcap_{i \in \mathcal{N}_j(\mathcal{G})} D\left(\frac{p_j(t+1) + p_i(t+1)}{2}, \frac{r-\sigma}{\sqrt{2}}\right)) \end{aligned}$$

The last content equality can be obtained as follows. In a general dimensional space, we can not say $E[D(\frac{p_j + \overline{p}^j_i}{2}, \frac{r-\sigma}{2})] \subseteq D(\frac{p_j + p_i}{2}, \frac{r-\sigma}{2})$, however $D(\frac{p_j+\overline{p}_i^j}{2}, \frac{r-\sigma}{2})$ is contained in a square A centered at $\frac{p_j+\overline{p}_i^j}{2}$ with side $\frac{r-\sigma}{2}$. Using a component wise projection it is easy to see that $E[A \,|\, t] \subseteq D(\frac{p_j(t+1)+p_i(t+1)}{2}, \frac{r-\sigma}{\sqrt{2}})$ for every $j \in \{1,\ldots,n\}$ and $i \in \mathcal{N}_j(\mathcal{G})$. This is valid for any of the circumcenter algorithms, variant 1, and proves that the set of agents reaches a ball of radius $\frac{r-\sigma}{\sqrt{2}}$. In the 1D case and for the (standard) Circumcenter Algorithm, variant 1, we have that:

$$p \in \bigcap_{j=1}^{n} \Big(\{p_j(t+1)\} \cup E[\overline{Q}_j(t)]\Big)$$
$$= \bigcap_{j=1}^{n} \Big(\{p_j(t+1)\} \cup [p^i_M(t) - \frac{r-\sigma}{2}, p^i_m(t) + \frac{r-\sigma}{2}]\Big)$$

Let $p_m = \min\{p_i \,|\, i \in \{1,\ldots,n\}\}$ and $p_M = \max\{p_i \,|\, i \in \{1,\ldots,n\}\}$. The above intersection is nonempty if and only if:

$$p = p_i \quad \text{for some } i \qquad \text{or} \quad p_M - p_m \leq r - \sigma\,. \quad (6)$$

Both conditions imply that we have reached a ball of diameter $r-\sigma$. In fact, if CC denotes the circumcenter of the set of all agents, we have that:

$$p_i \in D\Big(\text{CC}, \frac{r-\sigma}{2}\Big) \subseteq [p_M - \frac{r-\sigma}{2}, p_m + \frac{r-\sigma}{2}], \quad (7)$$

$\forall\, i \in \{1,\ldots,n\}$. This is valid for any graph $\mathcal{G}$.

Now consider the particular case of $\mathcal{G}_{\text{disk}}(r)$. Condition (6) implies that p_m and p_M are connected and we have reached the complete graph. From the set content (7) we also see that it will not be necessary to enforce the constraint in the Modified (standard) Circumcenter Algorithm, variant 1, since it automatically holds. Therefore we have that $p_i(t+1) = \text{CC}(\overline{\mathcal{M}}_i)$ and $p = E[\text{CC}(\overline{\mathcal{M}}_i)|\mathcal{P}] = \text{CC}$, $\forall\, i \in \{1,\ldots,n\}$. Since $\text{CC}(\overline{\mathcal{M}}_i) \in D\Big(\text{CC}(\mathcal{M}_i), \sigma\Big)$, then it must be that $p_i(t+1) \in D\Big(\text{CC}(p_1,\ldots,p_n), \sigma\Big)$, $\forall\, i \in \{1,\ldots,n\}$. A more careful analysis of the interesection of balls in 2D would allow us to get a better estimate of the practical stability ball. In simulations, it can be seen that indeed the ball has typically a smaller radius than $(r-\sigma)/\sqrt{2}$.

5 Simulations

Figure 2 shows a run of the Modified (standard) Circumcenter Algorithm, variant 2, for $\mathcal{G}_{\text{disk}}(r)$ in 2D, 15 agents and 300 time steps. Here $r = 6$ and $\sigma = 3$. The connectivity of the group of the 15 agents is shown in the left box of Figure 2 while its evolution is shown in the right box. As it can be seen here, the algorithm behaves even better than expected from the 1D analysis. There is a slight wandering of the practical stability ball. This behavior is representative of what we have seen in many repeated simulations with different initial conditions and relations r/σ.

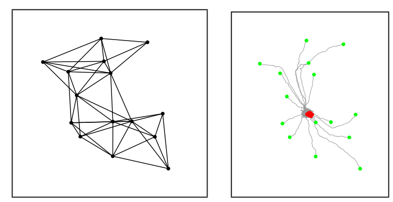

Fig. 2. Modified (standard) Circumenter Algorithm, variant 2, for 15 agents in 2D connected by $\mathcal{G}_{\text{disk}}(r)$, with $r = 3$. The noise error is bounded by $\sigma = 3$.

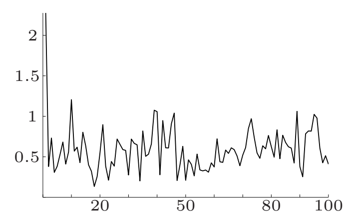

Fig. 3. Diameter evolution of a network of 4 agents in 1D connected by $\mathcal{G}_{\text{disk}}(r)$, $r = 3$, and evolving under the Modified (standard) Circumcenter Algorithm, variant 1. The noise error is bounded by $\sigma = 1$.

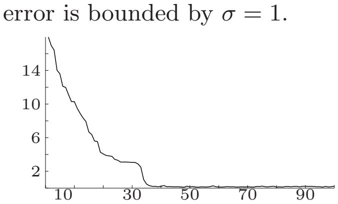

Fig. 4. Diameter evolution of a network of 50 agents in 1D connected by $\mathcal{G}_{\text{disk}}(r)$, $r = 3$, and evolving under the Modified (standard) Circumcenter Algorithm, variant 1. The noise error is bounded by $\sigma = 1$.

Simulations of the evolution of the diameter under the Modified (standard) Circumcenter Algorithm, variant 1, for $\mathcal{G}_{\text{disk}}(r)$ are shown in Figures 3 and 4 for a network of 4 and 50 agents. In general we observe that the smaller the group of agents, the increased wandering of the stability ball and the larger

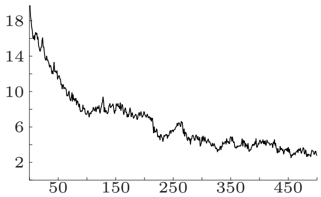

Fig. 5. Diameter evolution of a network of 30 agents in 1D connected by $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$, $r = 3$, and evolving under the Modified (standard) Circumcenter Algorithm, variant 1. The noise error is bounded by $\sigma = 1$ and the fixed graph $\mathcal{G}$ corresponds to the Limited Delaunay graph of the initial positions, see [5].

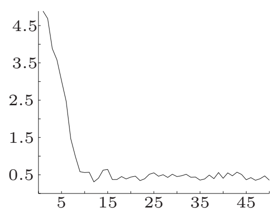

Fig. 6. Diameter evolution of a network of 15 agents in 2D connected by a sequence of graphs $\mathcal{G}(t) \subseteq \mathcal{G}_{\text{disk}}(r)$, $r = 2$, and evolving under the Modified 1/2 Circumcenter Algorithm, variant 1. The noise error is bounded by $\sigma = 1$ and $\mathcal{G}(t) = \mathcal{G}_{\text{disk}}(r)$ every 20 steps.

its diameter. When the number of agents is increased a filtering effect is produced which favors the final outcome. Note that, for the case of 50 agents, the diameter function remains constant for some period of time. This is due to the constraint enforcement that does not allow agents to move, which can happen when $r - \sigma$ is small. Since the information about the neighbors positions changes randomly in time according to a uniform distribution, it is clear that eventually the constraint sets will become nonempty and agents will be able to move. This is the essence of the proof of Theorem 12, which is actually proven for the Modified 1/2 Circumcenter Algorithm. A trick to diminish the wandering effect of the practical stability ball might be to pick an homogeneous $\sigma \leq \sigma_0 \approx r$ and $\sigma_0 < r$ in the final stages of the algorithm. A simulation of Modified (standard) Circumcenter Algorithm, variant 1, for a fixed graph $\mathcal{G} \subseteq \mathcal{G}_{\text{disk}}(r)$ and 30 agents is shown in Figure 5. Convergence here is much slower due to the fact that each agent has only two neighbors. Note also that the diameter may increase at any time, so a deterministic analysis like the for $\mathcal{G}_{\text{disk}}(r)$ is no longer feasible.

Finally, Figure 6 shows a simulation of the Modified 1/2 Circumcenter Algorithm, variant 1, for 15 agents connected in 2D under a sequence of switching graphs. Overall, we observe a decreasing trend until agents reach a ball of approximately $r - \sigma$ as a diameter.

6 Conclusions

In order to cope with measurement noise, to possible modifications of a class of circumcenter algorithms were proposed. Convergence is shown to happen deterministically (1D) and w.p.1 (2D) and the behavior is confirmed in several simulations.

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