Unicycle coverage control via hybrid modeling

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Abstract

This paper presents gradient-descent coverage algorithms for a group of nonholonomic vehicles. Similarly to previous approaches, the deployment strategy relies on Locational Optimization techniques and algorithms are distributed in the sense of the Delaunay graph. In order to deal with unicycle dynamics and guarantee performance, we introduce several vehicle modes and integrate them in a hybrid system We then analyze the algorithms with a recently introduced invariance principle for hybrid systems.

I. Introduction

The ability to autonomously deploy across a spatial region, as well as dynamically adjust to single-point failures gives mobile networks an advantage over static ones. This leads to the question of how to design effective motion coordination algorithms for their unsupervised control [1]. Due to the complexity that systems interacting over networks possess, it is reasonable to consider simple dynamical models for each vehicle in a first approximation. However, the nontrivial dynamics of current unmanned systems can invalidate the performance of the proposed algorithms. This work tries to contribute to this aspect by proposing a motion coordination strategy for the deployment of a nonholonomic mobile sensor network.

Although each robotic agent in a network may be controllable and the interaction among them can even be fixed, the consideration of non-trivial vehicle dynamics needs special treatment to avoid destabilizing effects. This has motivated a large number of papers on the design of coordination algorithms for multi-agent systems with fixed interaction topologies; see e.g., [2], [3], [4], [5] on formation stabilization and synchronization. In particular, the stability analysis of this class of algorithms can be approached via Lyapunov methods and the classical LaSalle

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invariance principle as from [6]. On the other hand, when the inter-vehicle interaction topology is not fixed, even the consideration of first-order integrator dynamics may require hybrid-systems or switched-systems techniques for analysis.

The use of multiple Lyapunov functions has been a predominant method for proving stability of a hybrid system, see [7], [8] and references therein. When dealing with multi-agent systems, however, much of our previous work, [9], [10], relied on LaSalle's invariance principle instead. The work in [11] provides a first extension of LaSalle's invariance principle to hybrid systems. More recently, the work in [12], [13] revisits the notion of hybrid (time) trajectories and develops a LaSalle invariance principle based on graphical convergence of set-valued maps. In this paper we choose the latter framework to present and analyze our system.

With respect to previous work, this paper contributes to current research on the control of nonholonomic vehicle networks. References include; e.g., obstacle avoidance [14], cyclic pursuit [15], [16], and path-planning for Dubins vehicles [17]. Here, we address a problem posed in an earlier work [9] regarding convergence of a coverage control problem using unicycle type dynamics. In [9] convergence to these configurations was proved for omni-directional vehicles. Wheeled vehicles were also considered, but the control algorithm was designed so that vehicles converged to a fixed target point as in [18], which was updated at discrete-time intervals. We lift this simplification allowing for target points (which depend on neighboring vehicles' positions) to vary continuously with time. This paper also presents an application of the results in [12] and how these can be useful in the context of multi-vehicle motion coordination. We refer the reader to [19] for an enlarged version of this manuscript, with all the proofs claims in the paper.

II. Problem setup and notation

In this section, we introduce the basic notation of the paper, some background on Locational Optimization, see [20], and a description of the unicycle vehicle dynamics that we consider.

We will denote by $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers, and $\mathbb{N}$ will be the set of non-negative integers. In the following, $(x, y, \theta) \in \mathrm{SE}_X(2)$ describes the position and orientation of a vehicle with respect to a fixed global coordinate frame, with $(x, y) \in X \subseteq \mathbb{R}^2$.

Let Q_0 be a convex polygon in $\mathbb{R}^2$ including its interior, and let $v \cdot w$ denote the inner product between $v, w \in \mathbb{R}^2$. Although we define Q_0 to be a convex polygon, for the sake of having a unique well-defined normal along the boundary ∂Q_0 we will replace the vertices of Q_0 with an arc of radius ϵ, where ϵ is arbitrarily small. Let Q denote the approximated Q_0. These “rounded corners” guarantee continuity of the following functions, facilitating the analysis later on. Let $t\colon \partial Q \to \mathbb{R}^2$ be the unit counterclockwise oriented tangent vector along the boundary of Q, and $t(x) = (t_1(x), t_2(x))$. We define the normal vector, $n\colon \partial Q \to \mathbb{R}^2$, as $n(x) = (-t_2(x), t_1(x))$, which points towards the interior of Q.

A. Locational Optimization

Let $\phi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a scalar field with bounded support Q. Here, ϕ represents an *a priori* measure of information on Q (the higher the value of $\phi(q)$ the more attention the group has to pay to q). Let $P = (p_1, \dots, p_n)$ be the *location of* n *sensors* in Q. We will consider the Locational Optimization [20] function:

$$\mathcal{H}(P, \mathcal{W}) = \sum_{i=1}^{n} \int_{W_i} \|q - p_i\|^2 \phi(q) dq \,, \tag{1}$$

where $\mathcal{W} = (W_1, \dots, W_n)$ is any partition of Q. The function (1) serves as a measure of how poor the coverage provided by the mobile sensing network in Q is. Smaller $\mathcal{H}$ has the interpretation of better coverage, thus we are interested in minimizing it. By introducing Voronoi partitions as in [10], the gradient of the cost function may be computed in a distributed fashion in the sense of the Delaunay graph. The ordinary Voronoi partition of Q is $\mathcal{V} = (V_1, \dots, V_n)$ where $V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall\, i \neq j\}$, $i \in \{1, \dots, n\}$. Each Voronoi region has mass M_{V_i} and centroid C_i, where

$$M_{V_i} = \int_{V_i} \phi(q) dq \,, \qquad C_i = \frac{1}{M_{V_i}} \int_{V_i} q\phi(q) dq \,.$$

III. Nonholonomic vehicle dynamics

The use of omni-directional vehicles in [9] allows the minimization of (1) via a Lloyd-like gradient descent control law. This control law forces individual agents to move directly towards the centroid of their Voronoi regions and is distributed in the sense of the Delaunay graph. That is, an agent only requires position knowledge of Delaunay neighbors to compute its own Voronoi region, and the corresponding centroid. We will introduce next several dynamical modes to guarantee that nonholonomic vehicles still propel toward these centroidal configurations.

The quantities that describe the vehicle are as follows. Referencing Figure 1, each vehicle has configuration variables $(p_i, \theta_i) \in \mathrm{SE}_Q(2)$, and a body coordinate frame with basis $e_{x_i} =$

$(\cos\theta_i, \sin\theta_i)$ and $e_{y_i} = (-\sin\theta_i, \cos\theta_i)$. We define the angle $\alpha_i \in [-\pi, \pi]$ to be the angle between e_{x_i} and a target point, in this case the region centroid C_i. As it will be clear later, in order to decrease $\mathcal{H}$, we will require $e_{x_i} \cdot (C_i - p_i) \geq 0$. In what follows, we denote $d_i = (C_i - p_i)$ as in Figure 1.

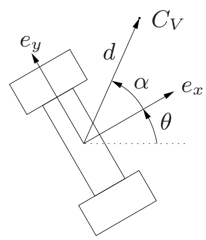

Fig. 1. Vehicle with wheeled mobile dynamics.

A. Variable forward velocity

Here we present the different dynamical modes under which vehicles in the network can evolve, and the intuition behind them. This will be made more formal in Section V.

Because the vehicle has control over both forward speed and turning rate, it can perform one of three maneuvers. A vehicle can move forward and steer, rotate in place, or be at a full stop. The dynamics that describe these motions are:

$$\dot{p}_i^1 = v\cos\theta_i\,, \quad \dot{p}_i^2 = v\sin\theta_i\,, \quad \dot{\theta}_i = \omega\,, \qquad \text{(forward)} \qquad (2)$$

$$\dot{p}_i^1 = 0\,, \quad \dot{p}_i^2 = 0\,, \quad \dot{\theta}_i = \omega\,, \qquad \text{(rotate)} \qquad (3)$$

$$\dot{p}_i^1 = 0\,, \quad \dot{p}_i^2 = 0\,, \quad \dot{\theta}_i = 0\,. \qquad \text{(rest)} \qquad (4)$$

An additional discrete variable, $l \in \{1, 2, 3\}$, will be used to describe which of the three modes (forward, rotation, and rest) a vehicle is in. Each agent can then be described by a state variable, $x_i \in \mathrm{SE}_Q(2) \times \{1, 2, 3\}$. The multiagent system state is denoted by $x = (x_1, \ldots, x_n) \in \mathbb{R}^{4n}$.

B. Vehicles with fixed forward velocity

Here we assume that each vehicle has a fixed forward velocity, $v = 1$, and a maximum turning velocity of ω_m. We also define the vehicle "virtual center" as its center of rotation when the turning input is $\pm\omega_m$. The location of these centers can be to the right or left of the vehicle, and we will introduce a switching strategy for vehicles that will avoid undesired Zeno effects.

The objective is to steer the virtual center of each vehicle to a desired centroid target. Once the virtual center has arrived at the centroid, the vehicle will "hover" about the centroid by maintaining the maximum steering input $\pm\omega_m$. We construct the dynamics of the virtual center by first assuming dynamics of the form $\dot{p}_i = \begin{pmatrix}\cos\theta_i, & \sin\theta_i\end{pmatrix}^T$, $\dot{\theta}_i = \omega_i$, where ω_i is the only input. Then the virtual center is located (in the local frame) at $\begin{pmatrix}0\,, & \pm\frac{1}{\omega_m}\end{pmatrix}^T$. We then transform this into the global frame and get $p_i' = p_i \pm \frac{1}{\omega_m}\begin{pmatrix}-\sin\theta_i, & \cos\theta_i\end{pmatrix}^T$. Now take the time derivative

$$\dot{p}_i' = \dot{p}_i \pm \frac{1}{\omega_m}\begin{pmatrix}-(\cos\theta_i)\dot{\theta}_i, & -(\sin\theta_i)\dot{\theta}_i\end{pmatrix}^T = \left(1 \mp \frac{\omega_i}{\omega_m}\right)\begin{pmatrix}\cos\theta_i, & \sin\theta_i\end{pmatrix}^T . \tag{5}$$

Indeed, with $\omega_i = \pm\omega_m$, the virtual center remains fixed.

IV. Hybrid automata review

Here we gather some useful results on the modeling and the stability analysis of hybrid automata. The exposition is taken from [13], [12] and included here for sake of completeness.

Definition 4.1 (Hybrid time domain): $D \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if $D = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$, for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in D$, $D \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid domain. •

Elements in hybrid time domains can be ordered: we say that $(t_i, j_i) \preceq (t_{i+1}, j_{i+1})$ iff $t_i + j_i \leq t_{i+1} + j_{i+1}$, $j \in \{1, \ldots, J\}$.

Definition 4.2 (Generalized solution): A generalized solution is a function $x(t, j) \in O$ defined on a hybrid time domain $\mathrm{dom}\, x$ such that: (1) on each interval $[t_j, t_{j+1}] \times \{j\} \subset \mathrm{dom}\, x$ of positive length (so that $t_j < t_{j+1}$) we have $\dot{x}(t, j) \in F(x(t, j))$, $x(t, j) \in A$, (2) for each $(t, j) \in \mathrm{dom}\, x$ such that $(t, j+1) \in \mathrm{dom}\, x$, we have $x(t, j+1) \in G(x(t, j))$, $x(t, j) \in B$. The set-valued maps $F : O \rightrightarrows \mathbb{R}^n$ and $G : O \rightrightarrows \mathbb{R}^n$ are the flow map and jump map, respectively. The sets $A \subset O$ and $B \subset O$ denote where the state may flow in continuous time, and where the state may make a discontinuous jump, respectively. It is possible for $A \cap B \neq \emptyset$, and in this case, both flowing and jumping may occur. Together, F, G, A, B define a hybrid system, $S = (F, G, A, B)$. •

Definition 4.3 (Weakly invariance): For a hybrid system $\mathcal{S}$, the set $\mathcal{M} \subset O$ is said to be:

(i) *weakly forward invariant* if for each $x_0 \in \mathcal{M}$ there exists at least one complete solution x with $x(t, j) \in \mathcal{M}$ for all $(t, j) \in \mathrm{dom}\, x$;

(ii) *weakly backward invariant* if for each $q \in \mathcal{M}, N > 0$ there exists $x_0 \in \mathcal{M}$ and at least one solution x such that for some $(t^*, j^*) \in \operatorname{dom} x$, $t^* + j^* > N, x(t^*, j^*) = q$ and $x(t, j) \in \mathcal{M}$ for all $(t, j) \preceq (t^*, j^*), (t, j) \in \operatorname{dom} x$;

(iii) *weakly invariant* if it is both weakly forward invariant and weakly backward invariant. •

Assumption 4.4 (Basic Conditions): A hybrid system $\mathcal{S} = (F, G, A, B)$ on a state space $O \subset \mathbb{R}^n$ satisfies the *Basic Conditions* if: (i) $O \subset \mathbb{R}^n$ is an open set, (ii) A and B are relative closed sets in O, (iii) F is outer semicontinuous, locally bounded on O, and convex $\forall\, x \in A$, (iv) G is outer semicontinuous, locally bounded on O, and satisfies $G(x) \subset O$, $\forall\, x \in B$. •

Theorem 4.5 (Hybrid LaSalle invariance principle, [12], (Corollary 4.3)): Given a hybrid system $\mathcal{S} = (F, G, A, B)$ that satisfies Assumption 4.4, suppose that: (i) $V : O \to \mathbb{R}$ is continuous on O and locally Lipschitz on a neighborhood of A, (ii) $\mathcal{U} \subset O$ is nonempty, (iii) $u_A(x) = \max_{f \in F(x)} \mathcal{L}_f V(x) \leq 0$ for all $x \in A$, (iv) $u_B(x) = \max_{x^+ \in G(x)} \{V(x^+) - V(x)\} \leq 0$ for all $x \in B$. Let x be precompact with $\overline{\operatorname{rge} x} \subset \mathcal{U}$. Then, for some constant $r \in V(\mathcal{U})$, x approaches the largest weakly invariant set in $V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_A^{-1}(0)} \cup u_B^{-1}(0)\right)$.

V. Vehicles with variable forward velocity

We refer the reader to the papers of Sanfelice et. al. [12], [13] for a review of the hybrid systems approach that follows. Also, an enlarged version of this manuscript can be found at [19].

A. Hybrid modeling

Here we formally define the hybrid system sketched in Section III-A, so that it satisfies the Basic Conditions in [12] in order to apply the invariance principle found therein. We will take the state-space of the entire system to be $O = \mathbb{R}^{4n}$, so that $x \in (\mathrm{SE}_Q(2) \times \{1, 2, 3\})^n \subset O$. We now define the hybrid system that models the nonholonomic vehicles, $\mathcal{S} = (F, G, A, B)$. In Section III-A, we described three different types of dynamics. Here we specify the relatively-closed set $A \subseteq O$, where continuous-time evolution occurs. To begin, we examine the configurations when a particular agent can flow, $A_i = A_i^1 \cup A_i^2 \cup A_i^3$:

$$\begin{aligned} A_i^1 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\},\, e_{x_i} \cdot d_i \geq \underline{\epsilon},\, \|d_i\| \geq \epsilon\}\,, \\ A_i^2 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{2\},\, \|d_i\| \geq \epsilon,\, e_{x_i} \cdot d_i \leq \epsilon\} \\ &\qquad \cup \{x \in O \mid x_i \in \mathrm{SE}_{\partial Q}(2) \times \{2\},\, \|d_i\| \geq \epsilon,\, e_{x_i} \cdot n \leq 0\}\,, \\ A_i^3 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{3\}, \|d_i\| \leq \overline{\epsilon}\}\,, \end{aligned}$$

where $0 < \underline{\epsilon} < \epsilon < \overline{\epsilon}$ and ϵ is arbitrarily small. Extending this to apply for all agents, we have

$$A = \bigcap_{i=1}^{n} A_i\,. \tag{6}$$

Since each $A_i^k \subset A_i$ is relatively closed in O, A is also relatively closed in O, satisfying one of the Basic Conditions see [19].

In forward motion, we propose a turning control gain $k_{\theta_i} < \infty$ proportional to the angular separation between the orientation of the vehicle and the target, α_i. Additionally, we will have a control gain $k_{p_i} < \infty$ that is proportional to the distance to the target. In rotation, we consider a constant turning rate of k_{θ_i}. We propose the following for each $i \in \{1, \ldots, n\}$:

$$\begin{aligned} F_i^1(x) &= \Big(\dot{p}_i^1, \quad \dot{p}_i^2, \quad \dot{\theta}_i, \quad \dot{l}_i\Big)^T = \Big(k_{p_i}\cos\theta, \quad k_{p_i}\sin\theta, \quad k_{\theta_i}\alpha_i, \quad 0\Big)^T\,, \\ F_i^2(x) &= \Big(0, \quad 0, \quad k_{\theta_i}\,\mathrm{sgn}(\alpha_i), \quad 0\Big)^T\,, \quad F_i^3(x) = \mathbf{0}\,. \end{aligned}$$

From here, we can define the flow map $F : O \rightrightarrows O$. When $x \notin A$, $F(x) = \emptyset$, and when $x \in A$,

$$F(x) = (F_1(x), \ldots, F_n(x))^T, \qquad F_i(x) = F_i^k(x) \iff l_i = k \in \{1, 2, 3\}\,. \tag{7}$$

We now define the sets of configurations, B_i, $i \in \{1, \ldots, n\}$, where a transition from flowing to jumping may occur for a particular agent. These cases are:

1) switching direction of travel, $B_i^1 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\}, e_{x_i} \cdot d_i \leq -\epsilon\}\,,$
2) forward motion to rotation (when the centroid is almost, perpendicular to the direction of travel or when the agent is on the boundary),
$$\begin{aligned} B_i^2 &= \{x \in O \mid x_i \in \mathrm{SE}_{\partial Q}(2) \times \{1\}, e_{x_i} \cdot n \leq -\epsilon\} \\ &\quad \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\}, -\epsilon \leq e_{x_i} \cdot d_i \leq \underline{\epsilon}\}\,, \end{aligned}$$
3) rotation to forward motion, $B_i^3 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{2\}, e_{x_i} \cdot d_i \geq \epsilon, e_{x_i} \cdot n \geq 0\}\,,$
4) forward motion or rotation to resting near a centroid,
$$B_i^4 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1, 2\}, \|d_i\| \leq \epsilon\}\,,$$

5) resting to forward motion, $B_i^5 = \{x \in O \mid x_i \in \text{SE}_Q(2) \times \{3\}, \|d_i\| \geq \overline{\epsilon}\}$,

where $\overline{\epsilon} > \epsilon$ and $\overline{\epsilon}$ is arbitrarily small. Then $B_i = \bigcup_{k=1}^{5} B_i^k$ and

$$B = \bigcup_{i=1}^{n} B_i \,. \tag{8}$$

It is not difficult to see that each B_i^k is relatively closed in O, and so B is also relatively closed in O, satisfying another Basic Condition, see [19]. A jump can occur if the state is in any of the five regions for a given $i \in \{1, \ldots, n\}$. The corresponding set of configurations, $G_i(x)$, where x might jump to are: $g_i^1(x) = (p_i, \theta_i + \pi, 1)$, $g_i^2(x) = (p_i, \theta_i, 2)$, $g_i^3(x) = (p_i, \theta_i, 1)$, $g_i^4(x) = (p_i, \theta_i, 3)$ and $g_i^5(x) = (p_i, \theta_i, 1)$. We combine the above functions for each vehicle and obtain $G_i(x) = \{(x_1, \ldots, g_i^k(x), \ldots, x_n) \mid x \in B_i^k, \forall\, k \in \{1, \ldots, 5\}\}$. The overall jump map $G : O \rightrightarrows O$ is $G(x) = \emptyset$, $x \notin B$, otherwise

$$G(x) = \bigcup_{i=1}^{n} G_i(x) \,. \tag{9}$$

Remark 5.1: The jump map G takes the state $x(t, j) \in B_i^k$ to another set, $x(t, j+1) \in A \cup B$. The following are all the possibilities: (1) If $k = 1$ then $G(x) \in A_i^1 \cup B_i^2 \cup B_i^4$, (2) If $k = 2$ then $G(x) \in A_i^2 \cup B_i^4$, (3) If $k = 3$ then $G(x) \in A_i^1 \cup B_i^4$, (4) If $k = 4$ then $G(x) \in A_i^3$, (5) If $k = 5$ then $G(x) \in A_i^1 \cup B_i^1 \cup B_i^2$. The state may also be in more than one jump set, such as $x \in B_i^2 \cup B_i^4$. When this happens, the state may jump according to $g_i^2(x)$ or $g_i^4(x)$, making this process non-deterministic. •

Remark 5.2: If we only implemented direction flipping, there exists a trajectory such that when $e_{x_i} \cdot d_i = 0$, the hybrid time domain (t, j) grows unbounded in j for fixed t. We include $\underline{\epsilon}, \epsilon, \overline{\epsilon}$, and the careful definition of A and B to prevent this and other similar situations. Other choices of the A, B sets are possible. •

Proposition 5.3: The hybrid system defined in equations (6), (7), (8), (9) satisfies the basic conditions of Assumption 4.4.

The proof can be found in the Appendix.

B. Asymptotic convergence

Our system satisfies the Basic Conditions, so we can apply the hybrid LaSalle invariance principle in [12]. We now state our main result.

Theorem 5.4: Let $\mathcal{U} = O$. Given the hybrid system defined in equations (6), (7), (8), (9), any precompact trajectory $x(t, j)$, with $\operatorname{rge} x \in \mathcal{U}$, will approach the set of points

$$\mathcal{M} = \{x \in O \mid x \in A_i^3, \forall\, i \in \{1, \ldots, n\}\} . \tag{10}$$

Proof: We choose V to be the cost function (1). It can be shown that (1) is locally Lipschitz on O [9]. For all x in A, $u_A(x) = \mathcal{L}_F \mathcal{H}$. We now compute the derivative, see [9].

$$\mathcal{L}_F \mathcal{H} = \sum_{i=1}^{n} \left[\frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial \mathcal{H}}{\partial \theta_i} \dot{\theta}_i + \frac{\partial \mathcal{H}}{\partial l_i} \dot{l}_i \right] = \sum_{i=1}^{n} \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i = \sum_{i=1}^{n} 2 M_{V_i} (p_i - C_i)^T \dot{p}_i \,.$$

When an agent is in a rotating or rest mode, $x_i \in A_i^2 \cup A_i^3$ and $\dot{p}_i = 0$. When $x_i \in A_i^1$, we have

$$\frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i = 2 M_{V_i} (p_i - C_i)^T \begin{pmatrix} k_{p_i} \cos \theta_i \\ k_{p_i} \sin \theta_i \end{pmatrix} = 2 k_{p_i} M_{V_i} (p_i - C_i) \cdot e_{x_i} \,.$$

Recall from the A_i^1 definition that $e_{x_i} \cdot (C_i - p_i) \geq \underline{\epsilon}$, then $\frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i < 0$. Thus, $u_A(x) \leq 0$, $\forall\, x \in A$.

Since G is set-valued, $u_B(x) = \max_{x^+ \in G(x)} \{\mathcal{H}(x^+) - \mathcal{H}(x)\}$. The cost function (1), does not have any dependence on l_i or θ_i. In addition, the jump map (9) does not create discontinuities in position. Thus, $\mathcal{H}$ does not change in value over jumps, and $u_B(x) = 0$, $\forall\, x \in B$.

The conditions of the hybrid LaSalle invariance principle are satisfied. Thus, the precompact trajectories x will approach the largest weakly invariant set in

$$L = V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_A^{-1}(0)} \cup u_B^{-1}(0) \right) = \mathcal{H}^{-1}(r) \cap \left(\overline{u_A^{-1}(0)} \cup B \right) ,$$

for some $r \in \mathcal{H}(\mathcal{U})$. Note that $\mathcal{H}^{-1}(r)$ represents some level set of the cost function (1). Now we must identify the largest weakly invariant set, $\mathcal{M}$ in L. Since our system is autonomous, the largest weakly forward invariant set is also the largest weakly invariant set.

We now check for weakly invariant trajectories. We do this by assuming that one vehicle is in a switching state, and show that it must switch to a flowing state, and remain there for a non-zero amount of time. Then we show that the only flowing state which remains in a level set for all time is the stationary state, $x \in A_i^3$ for all $i \in \{1, \ldots, n\}$.

Suppose there exists a trajectory $\tilde{x}(t, j)$ with $\mathcal{H}(\tilde{x}) = r$ for all $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $\tilde{x}(t_0, j_0) \in B$. This implies that there exists i^* and k^* such that $\tilde{x}(t_0, j_0) \in B_{i^*}^{k^*}$. Referencing Remark 5.1, all jumps eventually terminate with $\tilde{x}(t_0, j) \in A_i = A_i^1 \cup A_i^2 \cup A_i^3$ for some $j > j_0$. Furthermore, this configuration $\tilde{x}(t_0, j)$ remains in A_i for a non-trivial amount of time. We have shown that all configurations $x \in B$ return to flowing states. Now we examine the case where $\tilde{x}(t, j) \in A$ to arrive at the final result.

Suppose there exists a trajectory $\tilde{x}(t,j)$ with $\mathcal{H}(\tilde{x}) = r$ for all $(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ and $\tilde{x}(t_0, j_0) \in A$ for some $t_0 + j_0 \geq 0$. Since $\frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i < 0$ for any $x \in A_i^1$, this implies that $\tilde{x}(t_0, j_0) \in A_i^2 \cup A_i^3$ for all $i \in \{1, \ldots, n\}$. If this is true, then $\dot{p}_i = 0$ for all $i \in \{1, \ldots, n\}$. Suppose there exists an i^* such that $\tilde{x}(t_0, j_0) \in A_{i^*}^2$. Because $\dot{p}_i = 0$, C_i is constant for all $i \in \{1, \ldots, n\}$, and under the flow F_i^2, $|\alpha_{i^*}|$ decreases. Then, for some t_1, such that $t_0 \leq t_1 < \infty$, $\tilde{x}(t_1, j_0) \in B_{i^*}^3$ where a jumped is forced so that $\tilde{x}(t_1, j_0 + 1) \in A_{i^*}^1$. This implies that $u_A(\tilde{x}) < 0$, and the trajectory $\tilde{x}(t,j)$ leaves the level set $\mathcal{H}^{-1}(r)$.

Therefore, in order to remain in the level set $\mathcal{H}^{-1}(r)$, trajectories $x(t,j)$ must satisfy $x \in A_i^3$ for all $i \in \{1, \ldots, n\}$. This also satisfies $x \in \overline{u_A^{-1}(0)}$. ■

VI. Vehicles with fixed forward velocity

A. Virtual center switching

It is not difficult to show that the following holds for $\mathcal{H}$; see [19].

Lemma 6.1: Let $P = (p_1, \ldots, p_i, \ldots, p_n)$, and let $\tilde{P} = (p_1, \ldots, \tilde{p}_i, \ldots, p_n)$ where $\tilde{p}_i$ is closer to the centroid C_i, $\|\tilde{p}_i - C_i\| \leq \|p_i - C_i\|$. Let $\mathcal{V}(P)$ be the Voronoi partition of Q associated with P. Then,

$$\mathcal{H}(\tilde{P}, \mathcal{V}) \leq \mathcal{H}(P, \mathcal{V})\,. \tag{11}$$

$$\Delta\mathcal{H} = \mathcal{H}(\tilde{P}, \mathcal{V}) - \mathcal{H}(P, \mathcal{V}) = M_i(\|p_i - C_i\|^2 - \|\tilde{p}_i - C_i\|^2) \leq 0\,. \tag{12}$$

Additionally, $\mathcal{H}(\tilde{P}, \tilde{\mathcal{V}}) - \mathcal{H}(P, \mathcal{V}) \leq \Delta\mathcal{H}$ •

Proof: At first, we have

$$\mathcal{H}(P, \mathcal{W}) = \sum_{i=1}^{n} \int_{W_i} \|q - p_i\|^2 \phi(q) dq\,.$$

Note that the above expression is the sum of the moment of inertias of each region about the positioins p_i of the vehicles. Recalling the Parallel Axis Theorem, the moment of inertia of an object about any axis rotation parallel to an axis passing through the center of mass can be written as

$$I = I_{CM} + MR^2\,,$$

where I_{CM} is the moment of inertia about the axis through the center of mass, M is the mass of the object, and R is the perpendicular distance between the new axis and the axis trough the center of mass. Therefore, we can rewrite the cost function as

$$\mathcal{H}(P,\mathcal{W}) = \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} I_{i,CM} + M_i\|p_i - C_i\|^2 \, .$$

and then,

$$\mathcal{H}(\tilde{P},\mathcal{W}) = \mathcal{H}(P,\mathcal{W}) - M_i(\|p_i - C_i\|^2 - \|\tilde{p}_i - C_i\|^2) \, .$$

Because $\|\tilde{p}_i - C_i\| \leq \|p_i - C_i\|$, the main result follows. ■

We propose that each vehicle can switch the position of its virtual center only if the resulting improvement given by (12) is better than some threshold β, which implies that the actual improvement in cost is $\mathcal{H}(P,\mathcal{V}) - \mathcal{H}(\tilde{P},\tilde{\mathcal{V}}) \geq \beta$. In the next section we precisely define the hybrid vehicle modes and switching criteria.

B. Hybrid modeling

Each vehicle can have its virtual center located to the right or left of its direction of travel. Additionally, each vehicle can either be in "forward" motion or "hovering" motion. This results in four possible modes of operation for each vehicle: forward-left, hover-left, forward-right, and hover-right. We can enumerate each mode with the state $l_i \in \{1,2,3,4\}$, and each vehicle can be described by the following tuple $x_i = (p_i, \theta_i, l_i) \in \mathrm{SE}_Q(2) \times \{1,2,3,4\}$, where p_i is the location of the current virtual center, θ_i is the orientation of the virtual center, and l_i is the mode of operation. To simplify notation, let the opposite virtual center be $\tilde{p}_i = p_i \pm \frac{2}{\omega_m} e_{y,i}$, let $\tilde{d}_i$ denote the vector $C_i - \tilde{p}_i$ and let $\tilde{\alpha}_i$ denote the angle between $e_{x,i}$ and $\tilde{d}_i$, see Figure 2.

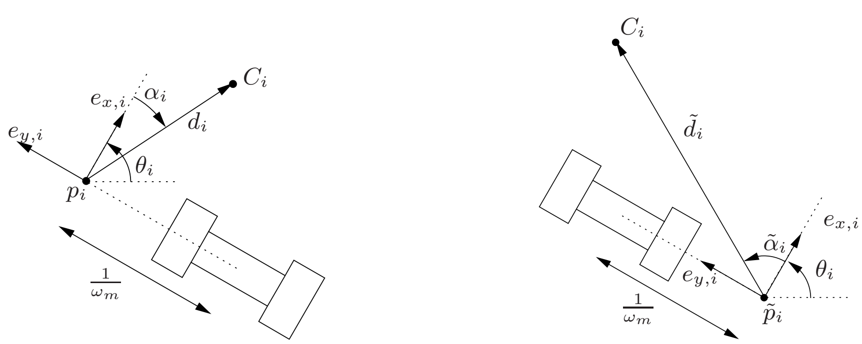

Fig. 2. Vehicle currently with a left virtual center, and the right virtual center configuration is also shown.

Following [12], we define state-space $O = \mathbb{R}^{4n}$. The flowing domain A is the subset of the state-space where continuous-time flow can occur. Let $A_i^1, \ldots, A_i^4$ be the set of points in O where a vehicle i can flow continuously in each of the four modes. We present a brief description of these sets followed by a precise set definition.

(1) An individual vehicle can be in A_i^1 (resp. A_i^3) if the centroid is in front of the left (resp. right) virtual center at p_i, and if p_i is not sufficiently close to C_i. Additionally, the improvement from switching between forward-left to forward-right (resp. vice-versa) given by (12) must be better than β. However, if the opposite virtual center $\tilde{p}_i$ is not in Q, then the vehicle may maintain its current virtual center despite violating the improvement threshold β.

$$\begin{aligned} A_i^1 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\},\, e_{x,i} \cdot d_i \geq \underline{\epsilon},\, M_i\|d_i\|^2 - M_i\|\tilde{d}_i\|^2 \leq \beta,\, \|d_i\| \geq \epsilon\} \\ &\quad \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\},\, e_{x,i} \cdot d_i \geq \underline{\epsilon},\, \tilde{p}_i \in \overline{Q^c},\, \|d_i\| \geq \epsilon\}\,, \\ A_i^3 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{3\},\, e_{x,i} \cdot d_i \geq \epsilon,\, M_i\|d_i\|^2 - M_i\|\tilde{d}_i\|^2 \leq \beta,\, \|d_i\| \geq \epsilon\} \\ &\quad \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{3\},\, e_{x,i} \cdot d_i \geq \underline{\epsilon},\, \tilde{p}_i \in \overline{Q^c},\, \|d_i\| \geq \epsilon\}\,, \end{aligned}$$

(2) A vehicle can be in A_i^2 (resp. A_i^4) if C_i is behind the left (resp. right) virtual center p_i, or if p_i is on the boundary Q and heading outwards, or if p_i is sufficiently close to C_i.

$$\begin{aligned} A_i^2 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{2\},\, e_{x,i} \cdot d_i \leq \epsilon,\, \|d_i\| \geq \epsilon\} \\ &\quad \cup \{x \in O \mid x_i \in \mathrm{SE}_{\partial Q}(2) \times \{2\},\, e_{x,i} \cdot n \leq 0\} \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{2\},\, \|d_i\| \leq \overline{\epsilon}\}, \\ A_i^4 &= \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{4\},\, e_{x,i} \cdot d_i \leq \epsilon,\, \|d_i\| \geq \epsilon\} \\ &\quad \cup \{x \in O \mid x_i \in \mathrm{SE}_{\partial Q}(2) \times \{4\},\, e_{x,i} \cdot n \leq 0\} \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{4\},\, \|d_i\| \leq \overline{\epsilon}\}\,. \end{aligned}$$

The use of hysteresis variables $0 < \underline{\epsilon} < \epsilon < \overline{\epsilon}$ serves to further insure that undesired Zeno effects do not occur. Combining these sets together, the entire system can flow if $x \in A$ where

$$A = \bigcap_{i=1}^{n} \left(A_i^1 \cup A_i^2 \cup A_i^3 \cup A_i^4\right)\,. \tag{13}$$

When the system is in the flowing domain A, the state evolves under the set-valued map F. Similar to the definition of A, we will present flow maps for individual vehicles and then compose them to form F. Let $F_i(x) = \begin{pmatrix}\dot{p}_i^1, & \dot{p}_i^2, & \dot{\theta}_i, & \dot{l}_i\end{pmatrix}^T$ with:

$$\begin{aligned} F_i^1(x) &= \begin{pmatrix}\cos\theta_i, & \sin\theta_i, & \frac{2\alpha_i\omega_m}{\pi}, & 0\end{pmatrix}^T, \quad F_i^2(x) = \begin{pmatrix}\cos\theta_i, & \sin\theta_i, & \omega_m, & 0\end{pmatrix}^T, \\ F_i^3(x) &= \begin{pmatrix}\cos\theta_i, & \sin\theta_i, & \frac{2\alpha_i\omega_m}{\pi}, & 0\end{pmatrix}^T, \quad F_i^4(x) = \begin{pmatrix}\cos\theta_i, & \sin\theta_i, & -\omega_m, & 0\end{pmatrix}^T, \end{aligned}$$

Then, for any $x \in A$,

$$F(x) = \Big(F_1(x), \quad \cdots, \quad F_n(x)\Big)^T, \qquad F_i(x) = F_i^k(x) \iff l_i = k \in \{1,2,3,4\}. \tag{14}$$

We now describe the subset of O where discrete jumps can occur. We will consider:

1) Switching from forward-left to forward-right,
$B_i^1 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\},\, e_{x,i} \cdot d_i \geq \underline{\epsilon},\, M_i(\|d_i\|^2 - \|\tilde{d}_i\|^2) \geq \beta,\, \tilde{p}_i \in Q\}$,
2) Switching from forward-right to forward-left,
$B_i^2 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{3\},\, e_{x,i} \cdot d_i \geq \underline{\epsilon},\, M_i(\|d_i\|^2 - \|\tilde{d}_i\|^2) \geq \beta,\, \tilde{p}_i \in Q\}$,
3) Switching from forward-left to hover-left,

$$B_i^3 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\},\, e_{x,i} \cdot d_i \leq \underline{\epsilon}\} \cup \{x \in O \mid x_i \in \mathrm{SE}_{\partial Q}(2) \times \{1\},$$
$$e_{x,i} \cdot n \leq -\epsilon\} \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{1\},\, \|d_i\| \leq \epsilon\},$$

4) Switching from hover-left to forward-left,
$B_i^4 = \{x \in O \mid x_i \in SE_Q(2) \times \{2\},\, e_{x,i} \cdot d_i \geq \epsilon,\, e_{x,i} \cdot n \geq 0,\, \|d_i\| \geq \overline{\epsilon}\}$
5) Switching from forward-right to hover-right,

$$B_i^5 = \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{3\},\, e_{x,i} \cdot d_i \leq \underline{\epsilon}\} \cup \{x \in O \mid x_i \in \mathrm{SE}_{\partial Q}(2) \times \{3\},$$
$$e_{x,i} \cdot n \leq -\epsilon\} \cup \{x \in O \mid x_i \in \mathrm{SE}_Q(2) \times \{3\},\, \|d_i\| \leq \epsilon\},$$

6) Switching from hover-right to forward-right.
$B_i^6 = \{x \in O \mid x_i \in SE_Q(2) \times \{4\},\, e_{x,i} \cdot d_i \geq \epsilon,\, e_{x,i} \cdot n \geq 0,\, \|d_i\| \geq \overline{\epsilon}\}$

The entire system can be in a jumping configuration if any one vehicle can jump. Therefore,

$$B = \bigcup_{i=1}^{n} \bigcup_{j=1}^{6} B_i^j. \tag{15}$$

With the switching domain defined, we present the jump map G. Let $g_i^1(x), \ldots, g_i^6(x)$ be the maps for an individual vehicle i. These maps are: $g_i^1(x) = (p_i - \frac{2}{\omega_m} e_{y,i}, \theta_i, 3)$, $g_i^2(x) = (p_i + \frac{2}{\omega_m} e_{y,i}, \theta_i, 1)$, $g_i^3(x) = (p_i, \theta_i, 2)$, $g_i^4(x) = (p_i, \theta_i, 1)$, $g_i^5(x) = (p_i, \theta_i, 4)$, $g_i^6(x) = (p_i, \theta_i, 3)$. We combine the above functions for each vehicle and obtain $G_i(x) = \Big\{(x_1, \ldots, g_i^k(x), \ldots, x_n) \mid x \in \bigcup_{j=1}^{6} B_i^j\Big\}$. The complete set-valued jump map is then

$$G(x) = \bigcup_{i=1}^{n} G_i(x). \tag{16}$$

Proposition 6.2: The flow domain and map A, F and the jump domain and map B, G described in (13)–(16) satisfy the basic conditions.

The proof of this proposition is similar to that of Proposition 5.3.

C. Convergence

We now apply the hybrid LaSalle principle of [12].

Theorem 6.3: Let $\mathcal{U} = O$. Given the hybrid system described in equations (13)–(16) with virtual center dynamics (5), any precompact trajectory $x(t, j)$, with $\operatorname{rge} x \in \mathcal{U}$, will approach the set of points

$$\mathcal{M} = \{x \in O \mid \|C_i - p_i\| \leq \overline{\epsilon}, \forall\, i \in \{1, \ldots, n\}\}\,. \tag{17}$$

Proof: We choose to analyze the system using the cost function $\mathcal{H}(P, \mathcal{V})$ from (1) with P being the locations the virtual centers. It can be shown that $\mathcal{H}$ is locally Lipschitz on O [9].

For all $x \in A$, $u_A = \mathcal{L}_F\mathcal{H}$. It can be seen that $\mathcal{L}_F\mathcal{H} = \sum_{i=1}^n 2M_{V_i}(p_i - C_i)^T \cdot \left(1 \mp \frac{\omega_i}{\omega_m}\right) \begin{pmatrix}\cos\theta_i \\ \sin\theta_i\end{pmatrix}$. When an agent is in a hovering mode, A_i^2, A_i^4, $\omega_i = \pm\omega_m$ and the virtual center remains stationary, therefore $\dot{\mathcal{H}} = 0$. When an agent is in forward mode, we have $\dot{\mathcal{H}} = \sum_{i=1}^n 2M_{V_i}\left(1 \mp \frac{2\alpha_i}{\pi}\right)(p_i - C_i)^T \begin{pmatrix}\cos\theta_i \\ \sin\theta_i\end{pmatrix}$. Additionally, $(p_i - C_i) \cdot \begin{pmatrix}\cos\theta_i, & \sin\theta_i\end{pmatrix} = -d \cdot e_{x,i} = -\cos\alpha_i$. Thus, $\dot{\mathcal{H}} = \sum_{i=1}^n -2M_{V_i}\left(1 \mp \frac{2\alpha_i}{\pi}\right)\cos\alpha_i$. A vehicle can only be in forward mode if $\alpha_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, see (13). Therefore $\cos\alpha_i \in [0, 1)$ and $\frac{2\alpha_i}{\pi} \in (-1, 1)$ and $\dot{\mathcal{H}} \leq 0$ for all $x \in A$, and the inequality is strict if there is at least one vehicle in forward motion.

Since G is set-valued, $u_B(x) = \max_{x^+ \in G(x)}\{\mathcal{H}(x^+) - \mathcal{H}(x)\}$. The cost function (1), does not have any dependence on l_i or θ_i. Thus, $\mathcal{H}$ changes only if virtual center positions change. However, when an agent switches from $l_i = 1$ to $l_i = 3$ or vice-versa, lemma 6.1 insures that the difference $\mathcal{H}(\tilde{P}, \tilde{\mathcal{V}}) - \mathcal{H}(P, \mathcal{V}) \leq -\beta$. Therefore, for all discrete jumps with $x \in B$, $u_B(x) = \max_{x^+ \in G(x)}\{\mathcal{H}(x^+) - \mathcal{H}(x)\} \leq 0$.

All conditions of the hybrid LaSalle invariance principle have been satisfied. The precompact trajectories x will approach the largest weakly invariant set in

$$L = V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_A^{-1}(0)} \cup u_B^{-1}(0)\right) = V^{-1}(r) \cap \left[\bigcap_{i=1}^n (A_i^2 \cup A_i^4)\right] \cup \left[\bigcup_{i=1}^n (B_i^3 \cup \cdots \cup B_i^6)\right].$$

for some $r \in \mathcal{H}(\mathcal{U})$. Note that $\mathcal{H}^{-1}(r)$ represents some level set of the cost function (1). Thus, we confine our search for the largest weakly invariant set to L. We now check for weakly invariant trajectories. We do this by assuming that one vehicle is in a switching state, and show that it must switch to a flowing state, and remain there for a non-zero amount of time. Then we show that the only flowing state which remains in a level set for all time is the hovering state with $\|p_i - C_i\| \leq \overline{\epsilon}$ for all $i \in \{1, \ldots, n\}$.

Suppose there exists a trajectory $\tilde{x}(t, j)$ with $\mathcal{H}(\tilde{x}) = r$ for all $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $\tilde{x}(t_0, j_0) \in B$. This implies that there exists i^* and k^* such that $\tilde{x}(t_0, j_0) \in B_{i^*}^{k^*}$. We confine our analysis to the cases where $\tilde{x}(t_0, j_0) \in (B_i^3 \cup \cdots \cup B_i^6)$. Let $\tilde{x}^+$ denote the state that $\tilde{x}(t_0, j_0)$ jumps to. The following transitions are possible: (1) $\tilde{x} \in B_i^3 \mapsto \tilde{x}^+ \in A_i^2$, (2) $\tilde{x} \in B_i^4 \mapsto \tilde{x}^+ \in A_i^1 \cup B_i^1$, (3) $\tilde{x} \in B_i^5 \mapsto \tilde{x}^+ \in A_i^4$, (4) $\tilde{x} \in B_i^6 \mapsto \tilde{x}^+ \in A_i^3 \cup B_i^2$.

We note that for the jumps that could result with $x^+ \in B_i^1 \cup B_i^2$ (cases 2 and 4), the system must jump again, but this jump decreases the cost function. In other words, these jumps take the system outside of the set L. The remaining possibilities result with $x^+ \in A_i$. The only possible trajectories that remain in the set L are those that jump to flowing states, $x^+ \in A$. Specifically, $x^+ \in A_i^2 \cup A_i^4$ for all $i \in \{1, \ldots, n\}$. Now we examine this case to arrive at the final result.

Suppose that there exists a trajectory $\tilde{x}(t, j)$ with $\mathcal{H}(\tilde{x}) = r$ for all $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$. such that $\tilde{x}(t_0, j_0) \in A_i^2 \cup A_i^4$ for all $i \in \{1, \ldots, n\}$. An agent in A_i^2 (resp. A_i^4) can only jump to forward motion by being in B_i^4 (resp. B_i^6). Since all agents are rotating about their virtual centers, the locations of the centroids, C_i for all $i \in \{1, \ldots, n\}$, remains fixed. This implies that d_i, $i \in \{1, \ldots, n\}$, also remain fixed. If there exists one agent i^* such that $\|d_{i^*}\| \geq \overline{\epsilon}$, then a jump eventually occurs since all vehicles rotate about their virtual centers with constant angular velocity. The system configuration will be $\tilde{x}(t_1, j_0) \in B_{i^*}^4$ (resp. $\tilde{x}(t_1, j_0) \in B_{i^*}^6$) for some $t_1 \geq t_0$. The resulting jump necessarily results in $\tilde{x}^+ \in A_i^1 \cup A_i^3$, and the trajectory $\tilde{x}$ leaves the level set $\mathcal{H}(\tilde{x}) = r$.

Thus, the only weakly invariant set in L is exactly that of (17). ■

VII. Simulations

We simulate $n = 8$ unicycles in $Q \subset \mathbb{R}^2 = [0, 10] \times [0, 10]$. The density function, ϕ, is composed of 3 Gaussian distributions $\phi(q) = 0.05 + 3\left[e^{-\frac{\|q - r_1\|^2}{2}} + e^{-\frac{\|q - r_2\|^2}{2}} + e^{-\|q - r_3\|^2}\right]$, where $r_1 = (8, 2)$, $r_2 = (8, 4)$ and $r_3 = (3, 7)$. The agent positions and orientations were randomly distributed in the bottom left corner, $l_i = 1$ for all $i \in \{1, \ldots, n\}$. We chose the control gains to be $k_{\theta_i} = 5$ and $k_{p_i} = \text{sat}\,\|C_i - p_i\|$. Note that any positive k_{θ_i} and k_{p_i} will work. Figure 3 shows that the wheeled vehicles with variable forward velocity do in fact converge to near-centroidal configurations. We present the case where vehicles have a fixed forward velocity in Figure 4.

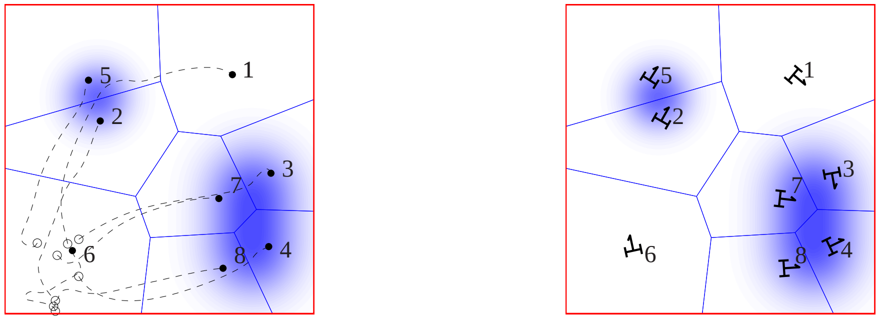

Fig. 3. Wheeled vehicle deployment simulation. The agents start in the lower left corner, denoted by the ‘o’. Path lines are shown in the left figure, and final positions and orientations in the right figure.

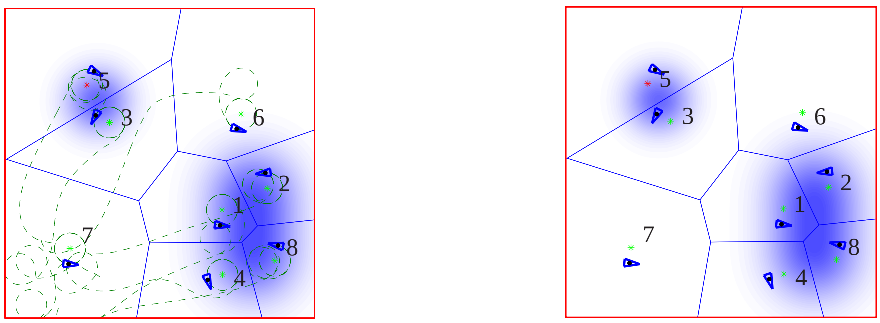

Fig. 4. Fixed forward velocity deployment simulation. The agents start in the lower left corner and path lines are shown in the left figure with final positions and orientations shown in the right figure. Virtual center locations are denoted by the star, ‘*’.

VIII. Acknowledgments

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IX. Appendix: Proof of Proposition 5.3

Here we show that the hybrid system defined in (6)–(9) satisfy the basic conditions of Assumption 4.4.

Proof: By construction, O is an open set, so basic condition (i) is true. In addition, each A_i^k, $k = \{1, 2, 3\}$, is closed since Q is a closed set, and the inequalities are continuous and closed. A is then relatively closed in O since A_i is the union of three closed sets, and A is the intersection of all A_i for $i \in \{1, \ldots, n\}$. For similar reasons, the set B_i for each $i \in \{1, \ldots, n\}$ is closed and this implies that B is relatively closed since it is a finite union of B_i for $i \in \{1, \ldots, n\}$. Thus, basic condition (ii) is true.

We now check basic condition (iii). The flow map F can map to a single point or to the empty set, both of which are convex. In addition, F is locally bounded because F_i^1, F_i^2, F_i^3, $i \in \{1, \ldots, n\}$ are bounded over $\mathbb{R}^{4n}$ given that k_θ and k_{p_i} are bounded.

Outer semicontinuity of F part 1, $x \in A$: It is important to note that each A_i, $i \in \{1, \ldots, n\}$ is the disjoint union, $A_i^1 \cup A_i^2 \cup A_i^3$. Suppose $x \in A_i^1$ for all $i \in \{1, \ldots, n\}$, the other cases where x is in one of A_i^2, A_i^3 are analogous. Consider now the convergent sequences $x_m \to x$ and $y_{i,m} \to y_i$ with $y_{i,m} = F_i^1(x_m)$ for all i.

(1) Suppose there exists an M such that $x_m \in A_i^1$ for all $m \geq M$. By continuity of F_i^1, $F_i^1(x_m)$ converges to $F_i^1(x)$. By unicity of limits, we have that $y_i = F_i^1(x)$ for all i.

(2) Suppose that for all $M \geq 0$ there exists $m_k \geq M$ such that $x_{m_k} \notin A_i^1$. Note that $\{x_{m_k}\} \subseteq \{x_m\}$. This implies that $x_{m_k} \in A_i^2 \cup A_i^3 \cup (O \setminus A_i)$. We can assume that x_{m_k} are all in one of these sets without loss of generality.

   a) Let $x_{m_k} \in A_i^2$ for all k. Since A_i^2 is closed, the limit of x_{m_k} is in A_i^2. This implies $x \in A_i^2$, a contradiction.

   b) Let $x_{m_k} \in A_i^3$ for all k. We will reach a similar contradiction.

   c) Let $x_{m_k} \in (O \setminus A)$ for all k. Then $y_{m_k} = F(x_{m_k}) = \emptyset$ for all k. Since the empty set is closed, $y_{m_k} \to y \in \emptyset$. Note that $\emptyset \subset F(x_i)$ and the result follows.

Outer semicontinuity of F part 2, $x \notin A$: If $x \notin A$, then $F(x) = \emptyset$. Suppose also that there exists convergent sequences $x_m \to x$ and $y_m \to y$ such that $y_m = F(x_m)$ for all m.

1) Assume that $x_m \notin A$ for all $m \geq M$. Then $F(x_m) = \emptyset$ and $y_m = \emptyset$ for all $m \geq M$. Since $\emptyset$ is closed, $y_m \to y \in \emptyset$.

2) Suppose there exists an infinite subsequence $\{x_{m_k}\} \subseteq \{x_m\}$ with $x_{m_k} \in A$ for all k. Since A is closed, $x_{m_k} \to x \in A$, a contradiction.

Finally, we prove basic condition (iv). The map G is strictly set-valued since a particular x_i can jump to multiple configurations, see (9). To prove local boundedness, consider an $x \in B$. We have to find a neighborhood $\mathcal{N} \subset O$ of x such that $\bigcup_{\bar{x} \in \mathcal{N}} G(\bar{x})$ is bounded. Observe that,

$$
\begin{aligned}
\bigcup_{\bar{x} \in \mathcal{N}} G(\bar{x}) &= \bigcup_{i, \bar{x} \in \mathcal{N}} G_i(\bar{x}) \\
&= \bigcup_{i=1}^{n} \{(x_1, \ldots, g_i^k(\bar{x}), \ldots, x_n) \mid \bar{x} \in \mathcal{N} \cap B_i^k\} \,.
\end{aligned}
$$

Each $g_i^k(x)$ is clearly locally bounded, so then we can find a $\mathcal{N}_0 \subset O$ of x such that $\bigcup_{\bar{x}\in\mathcal{N}_0} G(\bar{x})$ is a bounded set. Therefore, G is locally bounded. We now prove outer semicontinuity of G. Suppose there exist two convergent sequences $x_m \to x$ and $y_m \to y$ such that $y_m \in G(x_m)$. We must prove that $y \in G(x)$.

Outer semicontinuity of G part 1: Suppose that for all $K \geq 0$ there exists $m_k \geq K$ such that $x_{m_k} \notin B$. Then $G(x_{m_k}) = \emptyset$ and $y_{m_k} \in \emptyset$. Since the empty set is closed, $y_{m_k} \to y \in \emptyset$, and it is true that $\emptyset \subseteq G(x)$, for any $x \in O$.

Outer semicontinuity of G, part 2: Suppose that for all $K \geq 0$ there exists $m_k \geq K$ such that $x_{m_k} \in B$. Since B is closed, this implies $x_{m_k} \to x \in B$. If $x_{m_k} \in B$, this implies there exist fixed i_0 and k_0 such that $x_{m_k} \in B_{i_0}^{k_0}$ for an infinite number of m_k. Without loss of generality denote $\{x_{m_k}\}$ as $\{x_m\}$. Since each B_i^k are closed, then $x_m \in B_{i_0}^{k_0}$ and $x_m \to x \in B_{i_0}^{k_0}$. Now, $y_m = (x_1, \ldots, g_{i_0}^{k_0}(x_m), \ldots, x_n) \in G(x_m)$. Since $g_{i_0}^{k_0}(x_m)$ is continuous, $g_{i_0}^{k_0}(x_m) \to g_{i_0}^{k_0}(x)$. By unicity of limits, $y_m \to y \in G(x)$. ∎