Distributed interpolation schemes for field estimation by mobile sensor networks

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***Abstract*— We introduce a procedure to adapt local interpolations to represent spatial fields as they are measured by a mobile sensor network. The scheme incorporates new sensor (synchronous) measurements in a similar fashion to a Kalman filter-like recursion. We derive necessary conditions that allow the distributed computation of the recursion and present an algorithm that makes use of agreement rules that satisfy them. We show how the Nearest Neighbor interpolation scheme is compatible with the motion coordination algorithm for region coverage proposed in [1]. Finally, we illustrate the performance of the algorithms in simulation.**

I. Introduction

An intensive research activity is being directed to the development of coordination algorithms that allow the use of multi-vehicle sensor networks in practical scenarios. Examples of such systems used in exploration and scientific ventures include multi-buoy systems [2], coordinated gliders for oceanographic research [3], and unmanned aerial vehicles (UAVs) for atmospheric observation [4].

Typically, these sensor networks are required to communicate with a base station that gathers all the information needed to produce an approximation of the spatial fields being measured. This leads to a centralized control architecture which is not scalable to large numbers of vehicles, it is non-robust to station failures, and becomes too rigid to cope with changing conditions in the environment. In particular, the process of directing vehicles to time-varying cues can be significantly slowed down, since the processing of the data is delayed until all measurements are gathered at the base station. Placing part of the estimation and processing load on the vehicles themselves will allow for greater autonomy and increase the capacity of reaction much needed for adaptive sampling applications. In order to make this possible, the identification of suitable methods for cooperative estimation and conditions for their distributed computation should be investigated. An additional challenge is to produce estimation algorithms that are compatible with other multi-vehicle system tasks. As part of this effort, this paper presents a (non-parametric) inference method whose computation can be distributed and is compatible with coverage algorithms in [1].

Literature review. The investigation of the requirements needed for decentralized estimation dates back to the ’80; see e.g., [5], and is related to the area of sensor fusion. The synthesis of distributed coordination algorithms for multi-agent sensor systems is the subject of current research. In particular, agreement and consensus algorithms [6], [7] have been widely analyzed and proposed for sensor fusion [8], and as a way to decentralize Kalman filters [9]. The devise of optimal sensor placement or motion coordination plans have been recently addressed to improve Kalman-filter estimation procedures for target tracking [10], or the optimal sampling of spatial fields [3]. The assumption of fixed communication topologies, all-to-all communication or the existence of a central station that is able to fuse information and communicate with all vehicles is a restriction considered in most cases. A related paper to the present work is [11], which investigates user information retrieval protocols from a static sensor network based on a Nearest Neighbor partition of the space. However, [11] leaves the problem of sensor data fusion unaddressed. In [12] a distributed parametric estimation approach is is presented that makes use of consensus algorithms to agree on the parameter distribution. The paper [13] makes use of kriging techniques for the distributed estimate of the gradient of a random field.

Statement of contributions. We introduce a procedure to adapt local interpolations to represent spatial fields by a multi-vehicle sensor system. The interpolation provides a non-parametric estimate of the field, which is refined via a Kalman filter-like recursion as new measurements are collected. We derive the expression of the optimal gain of the filter and obtain conditions under which the scheme admits decentralization. For the case of Nearest Neighbor interpolations, the required inter-vehicle communication graph should contain a newly-identified proximity graph function that is related to the Delaunay graph. Finally, we discuss how the schemes can be modified to account for data compression procedures to make them more scalable.

II. Problem statement and preliminaries

Here we state the general problem scenario with given assumptions, and introduce basic preliminaries on Voronoi partitions, graphs and spatial interpolation methods; some references are [14], [15], [16].

A. Motivating Problem and Assumptions

Let $\mathbb{R}_{\geq 0}$ denote the positive real numbers including 0 and let $p_1, \dots, p_n$ denote the positions of n vehicles moving on a compact and convex region of the space $Q \subseteq \mathbb{R}^d$. We assume that each vehicle is endowed with physico-chemical sensors and is able to take point measurements z_i of certain scalar field $\phi : \mathbb{R} \times Q \to \mathbb{R}_{\geq 0}$. For example, ϕ might represent an environmental substance such as salinity concentration in the sea, aerosol pollutant in the atmosphere or any chemical concentration. For simplicity, we will consider here that ϕ is *static*; i.e., $\phi : Q \to \mathbb{R}_{\geq 0}$. We will also assume that the

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measurements z_i, $i \in \{1, \dots, n\}$, are affected by a spatially and temporally uncorrelated white noise. In other words,

$$z_i(t) = \phi(p_i(t)) + \epsilon_i(t)\,, \qquad \epsilon_i(t) \sim \mathcal{N}(0, \sigma)\,,\ \forall t \geq 0\,,$$
$$E[\epsilon_i(t)\epsilon_j(s)] = 0 \quad \text{for } i \neq j \text{ or } t \neq s\,,$$

where $\mathcal{N}(0, \sigma)$ is a zero-mean Gaussian distribution with covariance σ, and $E[\cdot]$ denotes the expectation operator. Under these assumptions, we would like to determine a distributed, non-parametric scheme for the collective and distributed estimation of ϕ. We shall assume that vehicles have access to their positions through; for example, GPS measurements.

B. Preliminaries on Graphs and Notation

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$ and let $B(p, R)$ denote the closed ball centered at p with radius R. A *graph* $\mathcal{G}$ is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined by a set of vertices $\mathcal{V} = \{1, \dots, n\}$, and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Graphically, a pair $(i, j) \in \mathcal{E}$ is represented by an arrow from vertex i to vertex j. The *adjacency matrix* of a graph, $A(\mathcal{G}) = (a_{ij}) \in \mathbb{R}^{n \times n}$, has entries $a_{ij} \in \{0, 1\}$ and $a_{ij} \neq 0$ iff $(i, j) \in \mathcal{E}$. The degree matrix of a graph, $D(\mathcal{G})$, is a diagonal matrix $D(\mathcal{G}) = \mathrm{diag}(d_1, \dots, d_n)$ such that $d_i = \sum_{j \neq i} a_{ij}$. The *set of neighbors* of vertex i in the graph $\mathcal{G}$ is denoted as $\mathcal{N}_i(\mathcal{G}) = \{j \in \mathcal{V} \,|\, (i, j) \in \mathcal{E}\}$. An *undirected graph* satisfies $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, while in a *directed graph* this property is not necessarily true.

A special class of graphs are *proximity graphs*. A proximity graph has as vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge set map $\mathcal{E} : \mathbb{R}^d \times \overset{(n)}{\cdots} \times \mathbb{R}^d \to 2^{\mathcal{V} \times \mathcal{V}}$ defined as a function of relative positions $\mathcal{P} = (p_1, \dots, p_n) \in \mathbb{R}^{dn}$ associated with the vertices in $\mathcal{V}$. That is, for each configuration $(p_1, \dots, p_n)$ a set of edges $\mathcal{E}(p_1, \dots, p_n) \subseteq \mathcal{V} \times \mathcal{V}$ is defined. An example of proximity graph is the *R-disk graph*, $\mathcal{G}_{R\text{-disk}} = (\mathcal{V}, \mathcal{E}_{R\text{-disk}})$, such that $(i, j) \in \mathcal{E}_{R\text{-disk}}(p_1, \dots, p_n)$ if and only if $\|p_i - p_j\| \leq R$. The set of neighbors of an agent in $\mathcal{G}_{R\text{-disk}}$ are those indices whose positions are contained in the ball centered at p_i with radius R. That is, $\mathcal{N}_i(\mathcal{G}_{R\text{-disk}}) = \{j \in \mathcal{V} \,|\, p_j \in B(p_i, R)\}$.

A proximity graph that we will use in the sequel is the Delaunay graph associated with a Voronoi partition. Recall that the (Euclidean) Voronoi partition of $Q \subseteq \mathbb{R}^d$ generated by a tuple of n distinct positions $\mathcal{P} = (p_1, \dots, p_n) \in \mathbb{R}^{nd}$ is a collection of sets $\mathscr{V}(\mathcal{P}) = \{V_i(\mathcal{P})\}_{i=1}^n$ such that $\cup_{i=1}^n V_i(\mathcal{P}) = Q$, and $V_i(\mathcal{P})$ is the region defined as:

$$V_i(\mathcal{P}) = \{q \in Q \,|\, \|q - p_i\| \leq \|q - p_j\| \text{ for all } j \neq i\}\,,$$

for all $i \in \{1, \dots, n\}$. We will usually refer to $V_i(\mathcal{P})$ as V_i. It is easy to see that p_i belongs to its Voronoi region V_i, $i \in \{1, \dots, n\}$. We say that j is a *Voronoi neighbor* of i if and only if V_i and V_j have a non-empty intersection at their boundary. Voronoi partitions give rise to the undirected *Delaunay graph*, $\mathcal{G}_{\mathrm{D}} = (\mathcal{V}, \mathcal{E}_{\mathrm{D}})$, defined over the set of vertices $\mathcal{V} = \{1, \dots, n\}$ and edge set $\mathcal{E}_{\mathrm{D}} \equiv \mathcal{E}_{\mathrm{D}}(p_1, \dots, p_n) = \{(i, j) \,|\, V_i \cap V_j \neq \emptyset,\ j \in \{1, \dots, n\}\}$. We refer to [14] for details on Voronoi partitions and the Delaunay graph. We will denote the set of Voronoi neighbors of p_i as either $\mathcal{N}_i = \{j \in \{1, \dots, n\} \,|\, V_i \cap V_j \neq \emptyset\}$ or $\mathcal{N}_{p_i}$, when there is no risk of confusion.

Assume that a sequence of configurations $\{\mathcal{P}^\ell = (p_1^\ell, \dots, p_n^\ell)\}_{\ell \in \mathbb{N}} \subseteq Q$ is defined over a time schedule $\mathbb{Z}_{\geq 0}$. Denote by $\Sigma_{\mathcal{G}}$ the set of graphs over the set of vertices $\{1, \dots, n\}$, including proximity graphs. We define the Nearest Neighbor graph map, $\mathcal{G}_{\mathrm{NN}} : \mathbb{Z}_{\geq 0} \to \Sigma_{\mathcal{G}}$, such that for each $\ell \in \mathbb{Z}_{\geq 0}$, $\mathcal{G}_{\mathrm{NN}}(\ell)$ is the directed graph $\mathcal{G}_{\mathrm{NN}}(\ell) = (\mathcal{V}, \mathcal{E}_{\mathrm{NN}}(\ell))$, with $\mathcal{N}_i(\mathcal{G}_{\mathrm{NN}}(\ell)) = \{j \in \mathcal{V} \mid V_j(\mathcal{P}^{\ell-1}) \cap V_i(\mathcal{P}^\ell) \neq \emptyset\}$. We will use the shorthand notations $V_i(\mathcal{P}^\ell) = V_i^\ell$ and $\mathcal{N}_i^{\mathrm{NN}}(\ell) = \mathcal{N}_i(\mathcal{G}_{\mathrm{NN}}(\ell))$, $i \in \{1, \dots, n\}$, $\ell \in \mathbb{Z}_{\geq 0}$, from now on. Similarly, we define the Local Inverse Distance Weighting (LIDW) graph map, $\mathcal{G}_{\mathrm{LIDW}} : \mathbb{Z}_{\geq 0} \to \Sigma_{\mathcal{G}}$, such that $\mathcal{G}_{\mathrm{LIDW}}(\ell) = (\mathcal{V}, \mathcal{E}_{\mathrm{LIDW}}(\ell))$, with $\mathcal{N}_i(\mathcal{G}_{\mathrm{LIDW}}(\ell)) = \{j \in \mathcal{V} \,|\, V_j^{\ell-1} \cap B(p_j^{\ell-1}, R) \cap V_i^\ell \cap B(p_j^\ell, R) \neq \emptyset\}$. In the sequel, we will use the notation $\mathcal{N}_i^{\mathrm{LIDW}}(\ell) = \mathcal{N}_i(\mathcal{G}_{\mathrm{LIDW}}(\ell))$, for $i \in \{1, \dots, n\}$, $\ell \in \mathbb{Z}_{\geq 0}$.

C. Spatial Interpolation Methods

There are several methods available to predict multi-variate fields $\phi : Q \to \mathbb{R}_{\geq 0}$ from scattered data. One of such approaches relies on spatial interpolations to provide non-parametric estimates of fields. In the absence of measurement noise, the general formulation of a spatial interpolation problem is the following: given the n values of the studied phenomenon, $z_i = \phi(p_i)$, measured at discrete points $p_1, \dots, p_n$, find a function $\Phi : Q \to \mathbb{R}_{\geq 0}$ such that $\Phi(p_i) = z_i$, for all $i \in \{1, \dots, n\}$.

An interpolant Φ is called *global* (resp. *local*), when the value of Φ at any point $q \in Q$ depends on *all* the data values (resp. only on data values at *"nearby"* points). Global interpolants are affected by the addition or deletion of data values and by changes in the location of data sites, while local interpolants are only affected at a vicinity of the changes. The required scalability properties of distributed systems and their decentralized nature make local interpolants more readily adaptable for groups of multiple vehicles.

Some of the most widely used local interpolation methods include the Nearest Neighbor (NN) and Natural Neighbor (Nat) interpolations, and interpolations based on a Triangulated Irregular Networks (TIN) [17], [14], [16]. The simplest interpolation of a function over Q is given by the Nearest Neighbor (NN) rule:

$$\Phi(q) = z_i\,, \qquad \|q - p_i\| < \|q - p_j\|, \quad j \neq i\,.$$

The resulting function is discontinuous at the boundaries of the Voronoi regions $V_i(\mathcal{P})$ associated with the location of the p_i, $i \in \{1, \dots, n\}$. An extension of this method is given by the Natural Neighbors interpolation method defined as follows. Given a point $q \in Q$ and a set of locations $\mathcal{P}$, compute $\mathscr{V}(\mathcal{P} \cup \{q\})$. The value $\Phi(q)$ becomes:

$$\Phi(q) = \sum_{i \in \mathcal{N}_q} w_i(q) z_i\,, \quad w_i(q) = \frac{\lambda_i(q)}{\sum_{k \in \mathcal{N}_q} \lambda_k(q)},$$

for $i \in \{1, \dots, n\}$. Here, $\mathcal{N}_q$ denotes the set of neighbors of q in the Delaunay graph associated with $\mathscr{V}(\mathcal{P} \cup \{q\})$ and, usually, the functions $\lambda_i(q)$ are chosen to be a function of the distance between q and p_i, $i \in \mathcal{N}_q$.

The Inverse Distance Weighting (IDW) interpolation method is a global interpolation defined as:

$$\Phi(q) = \sum_{i=1}^{n} w_i(q) z_i \,, \quad w_i(q) = \frac{\frac{1}{\|q-p_i\|}}{\sum_{k=1}^{n} \frac{1}{\|q-p_k\|}} \,,$$

for $i \in \{1, \dots, n\}$. A local version of IDW becomes:

$$\Phi(q) = \sum_{i=1}^{n} w_i(q) z_i, \; w_i(q) = \frac{\frac{1}{\|q-p_i\|} 1_{[0,R]}(\|q-p_i\|)}{\sum_{k=1}^{n} \left(\frac{1}{\|q-p_k\|} 1_{[0,R]}(\|q-p_k\|) \right)}$$

for $i \in \{1, \dots, n\}$. Here, $1_{[0,R]}(r)$ is the indicator function over the interval $[0, R]$. In other words, in this local version only the nodes p_i which are within distance R of q will affect the value of the interpolation.

Although the NN, Nat, and local IDW (LIWD) approaches do not give rise to continuous representations, they are computationally very fast and can be easily extended to any bounded set of any dimension. In comparison, the TIN approaches require the computation of a set of generalized tetrahedra in $\mathbb{R}^n$, which can lead to complications when defining partitions of compact domains. A solution to deal with this problem, see [14], requires the placement of many nodes along the boundary of Q. In the following, we will pay attention to the NN and LIDW interpolations for their computational simplicity and to obtain different adaptive interpolation schemes to estimate a field ϕ.

III. Centralized interpolation schemes

In this section, we introduce the centralized interpolation schemes that will serve as a basis for the distributed inference schemes proposed later. The schemes make use of the NN and LIDW interpolation rules refined through a Kalman-like procedure. We will consider that the time schedule $\mathbb{Z}_{\geq 0}$ is known by each agent and synchronizes the taking of the n independent measurements $z_i(\ell)$, $i \in \{1, \dots, n\}$, $\ell \in \mathbb{Z}_{\geq 0}$ and actions described later. This is a reasonable assumption for static fields, where waiting time periods for all vehicles can be established.

Agent i's *dominance region* at time $\ell \in \mathbb{Z}_{\geq 0}$, D_i^ℓ, is defined to be $D_i^\ell = V_i^\ell$ or $D_i^\ell = V_i^\ell \cap B(p_i^\ell, R)$, where R is the radius of spatial correlation in the LIDW interpolation method. Consider the classes of functions:

$$\begin{aligned} \overline{\mathscr{C}} &= \{\overline{\psi} : \mathbb{R} \times Q \to \mathbb{R}_{>0} \,|\, \forall\, t \in \mathbb{R},\, q \mapsto \overline{\psi}(t,q) \text{ piece. cont.}\}, \\ \mathscr{C} &= \{\psi : \mathbb{R} \times Q \to \mathbb{R} \,|\, \exists\, \overline{\psi} \in \overline{\mathscr{C}} \text{ s.t. } \psi(t,q) \sim \mathcal{N}(\overline{\psi}(t,q), \sigma) \\ &\qquad \text{and } E[\psi(t,p)\psi(s,q)] = 0 \text{ for } t \neq s \text{ or } p \neq q\}\,. \end{aligned}$$

Associated with these, we can define an *observation operator*, $\mathcal{Q} : \mathbb{Z}_{\geq 0} \times Q^n \times \overline{\mathscr{C}} \to \mathscr{C}$, for a given interpolation method. That is, $\mathcal{Q}(t_\ell, \mathcal{P}^\ell, \overline{\psi}) \in \mathscr{C}$ is a new static spatial field defined as $\mathcal{Q}(\ell, \mathcal{P}^\ell, \overline{\psi})(q) = \sum_{i=1}^{n} w_i^\ell(q)(\overline{\psi}(\ell, p_i^\ell) + \epsilon(\ell, p_i^\ell))$, for all $q \in Q$. Here $\overline{\psi}(\ell, p_i^\ell) + \epsilon(\ell, p_i^\ell)$ is the measurement of $\overline{\psi} \in \overline{\mathscr{C}}$ that sensor at p_i^ℓ takes, where $\epsilon : \mathbb{R} \times Q \to \mathbb{R}$ is a white noise such that $\epsilon(t,p) \sim \mathcal{N}(0, \sigma)$, $E[\epsilon(t,p)\epsilon(s,q)] = 0$ for $t \neq s$ or $p \neq q$. The function $w_i^\ell(q)$ is the weight corresponding to one of the mentioned interpolation methods.

In other words, $\mathcal{Q}$ provides a *snapshot* of a given $\overline{\psi}$ according to measurements at vehicle sites $\mathcal{P}^\ell$ at time $\ell \in \mathbb{Z}_{\geq 0}$. For simplicity, we will use the notation $\mathcal{Q}_\ell \psi \equiv \mathcal{Q}(\ell, \mathcal{P}^\ell, \psi)$, whenever it is clear that the sites $\mathcal{P}^\ell$ correspond to the vehicles' positions at time $\ell \in \mathbb{Z}_{\geq 0}$. Similarly, $\epsilon_i^\ell = \epsilon(\ell, p_i^\ell)$, for $\ell \in \mathbb{Z}_{\geq 0}$ and $i \in \{1, \dots, n\}$. The expected value of $\mathcal{Q}(\ell, \mathcal{P}_\ell, \overline{\psi})$ will be denoted as $\overline{\mathcal{Q}}_\ell \overline{\psi} = \sum_{i=1}^{n} \overline{\psi}(\ell, p_i) w_i^\ell(q)$.

Let $\phi : Q \to \mathbb{R}_{\geq 0}$ be the static field we would like to estimate. Suppose that there is an initial field estimate ϕ_0 available. As new measurements are taken, we use an update rule inspired by a Kalman-like recursion to refine the interpolation. The convex combination $\phi_\ell = \phi_{\ell-1} + W_\ell(\mathcal{Q}_\ell \phi - \mathcal{Q}_\ell \phi_{\ell-1})$, $\quad \ell \geq 1$, yields an estimated value $\overline{\phi}_\ell = E[\phi_\ell]$ of the field ϕ at time ℓ. By induction, one can see that:

$$\overline{\phi}_\ell = \overline{\phi}_{\ell-1} + W_\ell(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})\,, \quad \ell \geq 1. \tag{1}$$

Here, W_ℓ plays the role of the gain at time $\ell \in \mathbb{Z}_{\geq 0}$. The combination (1) is a weighted sum of the predicted value of the field, $\overline{\phi}_{\ell-1}$, and the *measurement innovation*, $\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}$, where $\mathcal{Q}_\ell \phi$ is the new observation of ϕ and $\mathcal{Q}_\ell \overline{\phi}_\ell$ is the predicted observation. The rule (1) thus produces a weighted average of measurement values and is understood as a pointwise equality for all $q \in Q$. The new estimate ϕ_ℓ will be different from the previous one as long as $\mathcal{Q}_\ell \phi(q) \neq \mathcal{Q}_\ell \phi_{\ell-1}(q)$ for all $q \in Q$. The fact that $\mathcal{Q}_\ell \phi(q) = \mathcal{Q}_\ell \phi_{\ell-1}(q)$ means that $\phi(p_i^\ell)$ has to be equal to a linear combination of previous measurements for all ℓ, which is not be true in general. By the law of large numbers, sampling repeatedly at all possible locations will make (1) approach the value of ϕ.

Given $\psi \in \overline{\mathscr{C}}$ and an estimate $\hat{\psi} \in \mathscr{C}$ such that $E[\hat{\psi}] = \psi$, we define the mean minimum square error (MMSE):

$$\mathrm{MMSE}(\psi, \hat{\psi}) = \int_Q E[(\psi(q) - \hat{\psi}(q))^2]\, dq\,.$$

In the following, we obtain an expression for $\mathrm{MMSE}(\phi, \phi_\ell)$, $\ell \in \mathbb{N}$, in order to find the optimal value of the gains which minimize this error. For simplicity we will use the notation $\mathrm{MMSE}(\phi, \phi_\ell) \equiv \mathrm{MMSE}_\ell$, $\ell \in \mathbb{N}$.

Lemma 1: The next equalities hold for $\ell \in \mathbb{N}$, $q, p \in Q$:

$$\begin{aligned} &E[\phi_{\ell-1}(q)\, \mathcal{Q}_\ell \phi(p)] = \overline{\phi}_{\ell-1}(q)\, \overline{\mathcal{Q}}_\ell \phi(p)\,, \\ &E[\mathcal{Q}_\ell \phi(q)\, \mathcal{Q}_\ell \phi(p)] = \overline{\mathcal{Q}}_\ell \phi(q)\, \overline{\mathcal{Q}}_\ell \phi(p) + \sigma^2\,, \\ &E[\mathcal{Q}_\ell \phi(q)\, \mathcal{Q}_\ell \phi_{\ell-1}(p)] = \overline{\mathcal{Q}}_\ell \phi(q)\, \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}(p) + \sigma^2\,. \end{aligned}$$

Using these formulas, one can obtain the expression $E[\phi_\ell(q)\, \phi_\ell(p)] = \overline{\phi_\ell}(q)\, \overline{\phi}_\ell(p) + \sigma^2 \Pi_{s=1}^{\ell}(1 - W_s)^2$.

Proof: We omit the proof for brevity. ■

Lemma 2: The following equalities hold for all $\ell \in \mathbb{N}$:

$$\begin{aligned} &E[(\mathcal{Q}_\ell \phi)^2] = (\overline{\mathcal{Q}}_\ell \phi)^2 + \sigma^2\,, \\ &E[(\mathcal{Q}_\ell \phi_{\ell-1})^2] = (\overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})^2 + \sigma^2 \left(1 + \Pi_{s=1}^{\ell-1}(1 - W_s)^2\right)\,, \\ &E[\phi\, \mathcal{Q}_\ell \phi] = \phi\, \overline{\mathcal{Q}}_\ell \phi\,, \\ &E[\phi\, \mathcal{Q}_\ell \phi_{\ell-1}] = \phi\, \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}\,, \\ &E[\phi_{\ell-1}\, \mathcal{Q}_\ell \phi_{\ell-1}] = \overline{\phi}_{\ell-1}\, \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1} + \sigma^2 \Pi_{s=1}^{\ell-1}(1 - W_s)^2\,. \end{aligned}$$

Using these formulas and Lemma 1, it is possible to obtain the recursive expression for $\ell \geq 1$:

$$\begin{aligned}
&\mathrm{MMSE}_\ell = \mathrm{MMSE}_{\ell-1} + \\
&W_\ell^2\Big(\int_Q (\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})^2\, dq + \sigma^2 \Pi_{s=1}^{\ell-1}(1-W_s)^2 M_Q\Big) \\
&-2W_\ell\Big(\int_Q (\phi - \overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})\, dq \\
&\qquad + \sigma^2 \Pi_{s=1}^{\ell-1}(1-W_s)^2 M_Q\Big), \qquad (2)
\end{aligned}$$

where M_Q is the volume of Q, $M_Q = \int_Q dq$.

Proof: We omit the proof for brevity. ■

Theorem 3: Consider the NN interpolation method (resp. the LIDW interpolation method with correlation radius $R > 0$). Given previous values W_s, $s \in \{1, \dots, \ell-1\}$, the optimal gain W_ℓ^{NN} (resp. W_ℓ^{LIDW}) that guarantees $\mathrm{MMSE}_\ell \leq \mathrm{MMSE}_{\ell-1}$, for all $\ell \in \mathbb{N}$, is given by:

$$\frac{\sum_{i=1}^n (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell)) \int_{D_i^\ell} (\phi(q) - \overline{\phi}_{\ell-1}(q)) dq + C}{\sum_{i=1}^n (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell))^2 M_{D_i^\ell} + C},$$

(resp. $W_\ell^{\mathrm{LIDW}} = (\sum_{i=1}^n [\sum_{j \in \mathcal{N}_i(\mathcal{G}_{\text{2R-disk}})} (\phi(p_j^\ell) - \overline{\phi}_{\ell-1}(p_j^\ell)) \int_{D_i^\ell} w_j^\ell(q)(\phi(q) - \overline{\phi}_{\ell-1}(q)) dq] + C) / (\sum_{i=1}^n [\sum_{k,j \in \mathcal{N}_i(\mathcal{G}_{\text{2R-disk}})} (\phi(p_j^\ell) - \overline{\phi}_{\ell-1}(p_j^\ell))(\phi(p_k^\ell) - \overline{\phi}_{\ell-1}(p_k^\ell)) \int_{D_i^\ell} w_j^\ell(q) w_k^\ell(q) dq] + C)$) where $C = \sigma^2 \Pi_{s=1}^{\ell-1}(1-W_s)^2 M_Q$ and $M_{D_i^\ell} = \int_{D_i^\ell} dq$, $i \in \{1, \dots, n\}$.

Proof: Taking the partial derivative of MMSE_ℓ with respect to W_ℓ in (2) and equating this to zero, we obtain:

$$\begin{aligned}
\frac{\partial\, \mathrm{MMSE}_\ell}{\partial W_\ell} &= 2W_\ell\Big(\int_Q (\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})^2\, dq + C\Big) \\
&\quad - 2\int_Q (\phi - \overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})\, dq - C
\end{aligned}$$

The critical value of the gain, W_ℓ^*, is thus given by:

$$W_\ell^* = \frac{\int_Q (\phi - \overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}) dq + C}{\int_Q (\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1})^2\, dq + C},$$

which satisfies $\dfrac{\partial^2\, \mathrm{MMSE}_\ell}{\partial W_\ell^2}\Big|_{W_\ell^*} > 0$; thus W_ℓ^* is a local minimum. The particular observation operator and the interpolation method that we use will further determine the value of the gain. In the following we present the derivations for the more simple NN interpolation, being the computations for LIDW analogous. Using that $\mathscr{V}(\mathcal{P}^\ell)$ is a partition of Q:

$$\begin{aligned}
&\int_Q (\overline{\mathcal{Q}}_\ell \phi(q) - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}(q))^2\, dq = \\
&\qquad \sum_{i=1}^n \int_{D_i^\ell} (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell))^2 \cdot 1_{D_i^\ell}(q) dq = \\
&\qquad \sum_{i=1}^n (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell))^2 M_{D_i^\ell},
\end{aligned}$$

where we have used the fact that $1_{D_i^\ell}(q) \cdot 1_{D_j^\ell}(q)$ is identically zero for all $i \neq j$ except for a set of measure zero, $\partial D_i^\ell \cap \partial D_j^\ell$.

A similar computation leads to:

$$\begin{aligned}
&\int_Q (\phi - \overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell \phi - \overline{\mathcal{Q}}_\ell \overline{\phi}_{\ell-1}) \\
&\qquad = \sum_{i=1}^n (\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell)) \int_{D_i^\ell} (\phi(q) - \overline{\phi}_{\ell-1}(q)) dq .
\end{aligned}$$

Thus the claimed expression for W_ℓ^{NN} using a NN interpolation is obtained. ■

Remark 4: The computation of the optimal gain, W_ℓ^*, in a practical setting can be done only approximately. For example, precise knowledge of the value of the integral of ϕ over the dominance regions D_i^ℓ, $i \in \{1, \dots, n\}$, for both the NN interpolation and the LIDW interpolation (integral in the numerator) is required. With limited information about ϕ, each vehicle can only compute this value approximately through e.g., quadrature rules [18]. Suppose ϕ is locally Lipschitz over Q and let $\Omega \subseteq Q$ be a compact subset. A quadrature rule for the computation of $\int_\Omega \phi(q) dq$ is defined as:

$$\int_\Omega \phi(q) dq \approx \sum_{k=1}^m \phi(q_k) \cdot M_{A_k}, \qquad (3)$$

where $q_i \in \Omega$ and $\{A_k\}_{k=1}^m$ is a partition of Ω into convex sets associated with $(q_1, \dots, q_m) \in \Omega^m$. The subtraction of both terms in (3) can be bounded as follows:

$$\begin{aligned}
&\Big| \int_\Omega \phi(q) dq - \sum_{k=1}^m \phi(q_k) \cdot M_{A_k} \Big| \\
&\quad \leq \sum_{k=1}^m \int_{A_k} |\phi(q) - \phi(q_i)| dq \leq L \sum_{k=1}^m \int_{A_k} |q - q_i| dq,
\end{aligned}$$

where L is the Lipschitz constant of ϕ. When $k \to \infty$, $M_{A_k} \approx 0$, the above is a good approximation. For a finite number of measurements in Ω, it can be proven; see [18], that the quadrature is minimized for A_k Voronoi regions and $q_k \in A_k$ being at the *centroids* of these regions; i.e., $q_k = C_{A_k}$, with $C_{A_k} = \frac{1}{\int_{A_k} \phi(q) dq} \int_{A_k} q\phi(q) dq$, for all $k \in \{1, \dots, m\}$. Since Q is compact and ϕ is piecewise continuous, each vehicle could take the simple approximation $\int_{D_i^\ell} \phi(q) dq \approx \phi(C_{D_i^\ell}) M_{D_i^\ell}$ for the computation of W_ℓ^{NN}. In order to approximate $C_{D_i^\ell}$, the estimate $\overline{\phi}_{\ell-1}$ can be used. The gain obtained in this way, $\widehat{W}_\ell^{\mathrm{NN}}$ is an approximation of W_ℓ^{NN}. As more measurements of ϕ are stored by vehicles (e.g., possibly taken along a path from p_i^ℓ to $C_{D_i^\ell}$) the approximation will improve and we will have $\widehat{W}_\ell^{\mathrm{NN}} \to W_\ell^{\mathrm{NN}}$ as $\ell \to \infty$. This approximation of the gain (which is optimal in certain sense) will induce a particular motion control algorithm on the vehicles to the centroids of their dominance regions. As we will see later, this is compatible with the task of coverage presented in [1]. As for LIDW, a possibility is:

$$\begin{aligned}
&\int_{D_i^\ell} \phi(q) w_j^\ell(q) dq \approx \phi(C_{ij}^\ell) \int_{D_i^\ell} w_j^\ell(q) dq, \text{ with} \\
&C_{ij}^\ell = \frac{1}{\int_{D_i^\ell} w_j^\ell(q) \overline{\phi}_\ell(q) dq} \int_{D_i^\ell} q w_j^\ell(q) \overline{\phi}_\ell(q) dq,
\end{aligned}$$

since to approximate the integrals of $w_j(q)\phi(q)$, $j \in \mathcal{N}_i^{\mathrm{LIDW}}$, vehicles would need to take several measurements. A similar discussion with an approximation $\widehat{W}_\ell^{\mathrm{LIDW}}$ of W_ℓ^{LIDW} holds.

On the other hand, the values $\phi(p_i^\ell)$ are also required for the computation of W_ℓ^{NN}. Since the mean value of measurements z_i is assumed to be the true $\phi(p_i)$, $i \in \{1,\dots,n\}$, the value $\phi(p_i)$ can be approximated by each vehicle by z_i at the same position. We will assume this procedure is taking place whenever we refer to the fact that a vehicle computes $\phi(p_i)$. This introduces another approximation error into the computation of the gain, but does not affect the convergence of the filter as we see in simulations. •

IV. Distributed interpolation schemes

In this section, we describe a distributed implementation of the centralized schemes of the previous section. As we see below, the distributed computation is possible if agents are able to form a jointly connected communication network in order to agree on the values of the optimal gains. On the other hand, in order to update the scheme estimate, communication according to $\mathcal{G}_{\mathrm{NN}}$ or, resp. $\mathcal{G}_{\mathrm{LIDW}}$ will have to be granted. An assumption that makes this more feasible is that the sensor network is dense in the environment; see Remark 7, for a discussion of this fact. On the other hand, it is not necessary that each vehicle maintains a global representation of ϕ, but just a local one over its region $D_i^{\ell-1} \cup D_i^\ell$.

Let W_ℓ^* denote the optimal gain obtained in Theorem 3 either by the NN or LIDW methods. We will assume that at each step of the time schedule $\ell \in \mathbb{Z}_{\geq 0}$ several actions (measurement taking and computations) are performed by agents synchronously. Additionally the time slot $[\ell, \ell+1]$ will be divided into communication rounds that will allow sensors to update their estimate of the gain. The main algorithm and functions that are called inside it are described in pseudocode in Table IV for both the NN and the LIDW interpolation methods. We provide an informal description of the algorithm in what follows, and establish the conditions needed for its correctness after this.

> *[Informal description].* At time ℓ each agent maintains in memory the current estimate of the field over the current dominance region, until the field is updated through the following sequence of steps. First, each agent computes an approximated "centroids" according to formula. This will a motion path for each agent has to visit while taking new measurements according to Remark 4. After this, a new dominance region is computed by communicating with neighbors. This will be the region over which the new estimate of the field will be updated. Simultaneously, agents determine which are the set of neighbors they need to transmit the information of the field estimate to. Then, the initial condition for a gain computation subroutine is computed; see Thm 5. The gain computation is completed after several synchronous communication rounds with other agents. Finally, each agent sends information to the right set of neighbors and receives the required information to update the field in the new region.

In order to compute the gain, each interval $[\ell, \ell+1]$ can be divided into T time slots to establish a set of communication rounds between neighboring agents in $\mathcal{G} \subseteq \mathcal{G}_{\mathrm{cmm}}(\ell)$ as follows. Let $m \in \{1,\dots,T\}$ index the communication rounds $t_m \in (\ell, \ell+1)$. At each t_m an undirected communication graph is established. That is, $\mathcal{G}(t_m) = (\{1,\dots,n\}, \mathcal{E}(m))$ is undirected and i, j can exchange a message at time t_m if and only if $(i,j) \in \mathcal{E}(m)$.

Suppose that, by communicating with neighbors in $\mathcal{G} \supseteq \mathcal{G}_{\mathrm{D}}$ for the NN interpolation approach (or $\mathcal{G} \supseteq \mathcal{G}_{2\mathrm{R\text{-}disk}}$ for the LIDW), agent i is able to compute at ℓ:

$$
\begin{aligned}
&N_i^{\ell+1}(0) \qquad (4)\\
&= (\phi(p_i^{\ell+1}) - \overline{\phi}_\ell(p_i^{\ell+1}))(\phi(C_i^\ell) M_{D_i^{\ell+1}} - \int_{D_i^{\ell+1}} \overline{\phi}_\ell(q)dq) + C\,,\\
&(\text{resp. } N_i^{\ell+1}(0) = \sum_{j \in \mathcal{N}_i(\mathcal{G}_{2R\text{-disk}})} (\phi(p_j^{\ell+1}) - \overline{\phi}_\ell(p_j^{\ell+1}))[\phi(C_{ij}^\ell) M_{D_i^{\ell+1}}\\
&\qquad - \int_{D_i^{\ell+1}} w_j^{\ell+1}(q)\overline{\phi}_\ell(q)\,dq + C)]\ )
\end{aligned}
$$

$$
L_i^{\ell+1}(0) = (\phi(p_i^{\ell+1}) - \overline{\phi}_\ell(p_i^{\ell+1}))^2 M_{D_i^{\ell+1}} + C\,, \qquad (5)
$$

$$
\begin{aligned}
&(\text{resp. } L_i^{\ell+1}(0) = \sum_{k,j \in \mathcal{N}_i(\mathcal{G}_{2R\text{-disk}})} (\phi(p_j^{\ell+1}) - \overline{\phi}_\ell(p_j^{\ell+1})) \times \qquad (6)\\
&\times (\phi(p_k^{\ell+1}) - \overline{\phi}_\ell(p_k^{\ell+1})) \int_{D_i^{\ell+1}} w_j^{\ell+1}(q) w_k^{\ell+1}(q)\,dq + C)
\end{aligned}
$$

where $C = \sigma^2 M_Q \Pi_{s=1}^{\ell-1}(1 - W_s^*)^2$. Then, during the T communication rounds take place, agents can update the values of $N_i^\ell(m)$ and $D_i^\ell(m)$, $m \in \{1,\dots,T\}$, by e.g., using an averaging algorithm. In the particular case that $K_\ell \to \infty$ the approximated value of the gain will tend to the original $\widehat{W}_\ell^*$ as discussed next.

Theorem 5: Let $\ell \in \mathbb{Z}_{\geq 0}$ be fixed. Suppose that at time ℓ each agent is able to compute (4) and (5). Denote by $\mathcal{G}(m)$, $m \in \{1,\dots,T\}$ the communication graphs established at rounds $t_m \in (\ell, \ell+1)$. Define the agreement algorithms:

$$
\begin{aligned}
N_j^{\ell+1}(m+1) &= \sum_{i=1}^n F_j^i(m) N_i^{\ell+1}(m),\\
D_j^{\ell+1}(m+1) &= \sum_{i=1}^n F_j^i(m) D_i^{\ell+1}(m), \qquad (7)
\end{aligned}
$$

where $F(s)$ is the stochastic matrix $F(m) = (I + D(\mathcal{G}(m)))^{-1}(I + A(\mathcal{G}(m)))$. Suppose we let $T \to +\infty$. If there exists a $M > 0$ such that for any $m_0 \in \mathbb{N}$ the union $\cup_{m=m_0}^{m_0+M} \mathcal{G}(m)$ is connected, then $N_j^{\ell+1}(m)/D_j^{\ell+1}(m) \to \widehat{W}_{\ell+1}^*$ exponentially fast as $m \to +\infty$, for all $j \in \{1,\dots,n\}$.

Proof: The proof is a consequence of the convergence properties of agreement algorithms, see [6], [19]. For undirected graphs, the agreement limit values are given by $\frac{1}{n}\sum_{i=1}^n N_i^{\ell+1}(0)$ and $\frac{1}{n}\sum_{i=1}^n D_i^{\ell+1}(0)$, respectively. Therefore $N_j^{\ell+1}(m)/D_j^{\ell+1}(m)$ converges to $(\sum_{i=1}^n N_i^{\ell+1}(0))/(\sum_{i=1}^n D_i^{\ell+1}(0)) = \widehat{W}_{\ell+1}^*$ as $m \to +\infty$.

The exponential convergence nature of consensus algorithms is also a known fact, see [19], and it depends on the connectivity of the $\mathcal{G}(m)$ and the number of agents n. ■

Function **Field-Estimate** using NN interpolation (resp. using LIDW interpolation)
Requires: Common consensus running time T, possible computation of own positions p_i^ℓ, communication with others in $\mathcal{G}_{\mathrm{D}}$ and $\mathcal{G}_{\mathrm{NN}}(\ell)$ (resp. $\mathcal{G}_{2\mathrm{R}\text{-disk}}$ and $\mathcal{G}_{\mathrm{LIDW}}(\ell)$)
Initialization: $\phi_0(q)$, D_i^0, and the initial measurement z_i^0 are obtained.

For $\ell \geq 0$ agent i does the following:

1: Compute the approximated centroid $C_i^\ell = \frac{1}{\int_{D_i^\ell} \overline{\phi}_\ell(q)dq} \int_{D_i^\ell} q\overline{\phi}_{\ell-1}(q)dq$.
(resp. the approximated centroids $C_{ij}^\ell = \frac{1}{\int_{D_i^\ell} w_j^\ell(q)\overline{\phi}(q)dq} \int_{D_i^\ell} qw_j^\ell(q)\phi(q)dq$)
2: Move to $p_i^{\ell+1} = C_i^\ell$, take a measurement $z_i^{\ell+1} = \phi(p_i^{\ell+1}) + \epsilon_i^{\ell+1}$, and compute $D_i^{\ell+1}$
(resp. move through the C_{ij}^ℓ, take measurements $z_{ij}^\ell = \phi(C_{ij}^\ell) + \epsilon_{ij}^\ell$, set $p_i^{\ell+1}$ equal to the last visited centroid, and compute $D_i^{\ell+1}$)
3: Compute the approximation $\int_{D_i^{\ell+1}} \phi(q)dq \approx z_i^{\ell+1} M_{D_i^{\ell+1}}$ (resp. compute the approximation $\int_{D_i^{\ell+1}} \phi(q)w_j^\ell(q)dq \approx z_{ij}^\ell \int_{D_i^{\ell+1}} w_j^\ell(q)dq$)
4: Compute the numbers:

$$N_i^{\ell+1} = (z_i^{\ell+1} - \overline{\phi}_\ell(p_i^{\ell+1}))(z_i^{\ell+1} M_{D_i^{\ell+1}} - \int_{D_i^{\ell+1}} \overline{\phi}_\ell(q)dq) + C$$

$$(\text{resp. } N_i^{\ell+1} = \sum_{j\in\mathcal{N}_i(\mathcal{G}_{2R\text{-disk}})} (z_i^{\ell+1} - \overline{\phi}_\ell(p_j^{\ell+1})) \left[z_{ij}^\ell M_{D_i^{\ell+1}} - \int_{D_i^{\ell+1}} w_j^{\ell+1}(q)\overline{\phi}_\ell(q)\, dq + C) \right] + C\ )$$

$$L_i^{\ell+1} = (z_i^{\ell+1} - \overline{\phi}_\ell(p_i^{\ell+1}))^2 M_{D_i^{\ell+1}} + C$$

$$(\text{resp. } L_i^{\ell+1}(0) = \sum_{k,j\in\mathcal{N}_i(\mathcal{G}_{2R\text{-disk}})} (z_j^{\ell+1} - \overline{\phi}_\ell(p_j^{\ell+1}))(z_k^{\ell+1} - \overline{\phi}_\ell(p_k^{\ell+1})) \int_{D_i^{\ell+1}} w_j^{\ell+1}(q) w_k^{\ell+1}(q)\, dq + C\ )$$

5: Obtain the approximated gain $W_{\ell+1} = \mathbf{GainAlgo}(T, N_i^{\ell+1}, L_i^{\ell+1})$
6: Compute the update of ϕ in the new region $\phi_{\ell+1}^i = \mathbf{FieldUpdate}(W_{\ell+1}, \phi_\ell^i, D_i^{\ell+1}, D_i^\ell)$,

return $\phi^i(q)$

Function **GainAlgo** for NN interpolation (resp. for LIDW interpolation)
Requires: Common consensus running time T, periodic communication graph connectivity, initial values of variables $N_i(0)$, $L_i(0)$.

for $m \in \{1, \ldots, T\}$ agent i does the following:

1: $N_i(m) = \frac{1}{|\mathcal{N}_i(m-1)|+1} \left(N_i(m-1) + \sum_{j\in\mathcal{N}_i(m-1)} (N_j(m-1) - N_i(m-1)) \right)$
2: $L_i(m) = \frac{1}{|\mathcal{N}_i(m-1)|+1} \left(L_i(m-1) + \sum_{j\in\mathcal{N}_i(m-1)} (L_j(m-1) - L_i(m-1)) \right)$
3: If $m = T$, compute $W = \frac{N_i(T)}{L_i(T)}$.

end **for**,
return W

Function **FieldUpdate** for NN interpolation (resp. for LIDW interpolation)
Requires: Previous estimate in region ϕ_ℓ^i, region D_i^ℓ, current region $D_i^{\ell+1}$, gain $W_{\ell+1}$, measurement $z_i^{\ell+1}$, communication with neighbors in $\mathcal{G}_{\mathrm{NN}}(\ell)$ (resp. communication with neighbors in $\mathcal{G}_{\mathrm{LIDW}}(\ell)$)

1: Obtain $g^j(q) = f_\ell^j(q) 1_{D_j^\ell(q)}$ from $j \in \mathcal{N}_i^{\mathrm{NN}}(\ell)$ (resp. from $j \in \mathcal{N}_i^{\mathrm{LIDW}}(\ell)$)
2: Compute $f_{\ell+1}^i(q) = \sum_{j\in\mathcal{N}_i^{\mathrm{NN}}(\ell+1)} g^j(q) + W_{\ell+1}(z_i^{\ell+1} - \phi_\ell^i(p_i^{\ell+1}))$
(resp. $f_{\ell+1}^i(q) = \sum_{j\in\mathcal{N}_i^{\mathrm{LIDW}}(\ell+1)} g^j(q) + W_{\ell+1} w_i^{\ell+1}(q)(z_i^{\ell+1} - \phi_\ell^i(p_i^{\ell+1})))$

return $f_{\ell+1}^i(q) 1_{D_i^{\ell+1}}$

TABLE I

FUNCTIONS FOR THE DISTRIBUTED INTERPOLATION FILTER USING THE NN APPROACH.

A decentralization procedure that requires less memory is one where vehicles just have information of $\overline{\phi}_\ell$ on their dominance regions D_i^ℓ, $\ell \in \mathbb{N}$. In fact, this is at least necessary for each vehicle to compute $N_i^\ell(0)$ and $D_i^\ell(0)$, $\ell \in \mathbb{N}$. More precisely, the information needed to compute $\overline{\phi}_\ell(q)$ over D_i^ℓ is the following:

(i) To compute $D_i^\ell(0)$ in the NN scheme, agent i needs to know the position of neighbors $j \in \mathcal{N}_i(\mathcal{G}_{\mathrm{D}}(\ell))$ (resp. in the LIDW scheme, agent i needs to know the position of neighbors $j \in \mathcal{N}_i(\mathcal{G}_{2R\text{-disk}}(\ell))$),

(ii) to compute $\overline{\phi}_\ell(q)$ in the NN scheme, agent i needs to communicate with neighbors $j \in \mathcal{N}_i^{\mathrm{NN}}(\ell)$ (resp. in the LIDW scheme, agent i needs to communicate with neighbors $j \in \mathcal{N}_i^{\mathrm{LIDW}}(\ell)$.)

Now vehicles can thus compute $\phi_\ell(q)$ over D_i^ℓ as follows.

Theorem 6: Let $p_1, \ldots, p_n$ denote the positions of a vehicle network moving over a convex region $Q \subseteq \mathbb{R}^d$. Suppose vehicles can synchronously take new measurements of a field $\phi : Q \longrightarrow \mathbb{R}$ as specified by the time schedule $\mathbb{Z}_{\geq 0}$. Assume also that agents can compute the gains W_ℓ^*, $\ell \in \mathbb{Z}_{\geq 0}$, in a distributed manner, e.g., as described in Theorem 5, and that $\{D_1^\ell, \ldots, D_n^\ell\}$ is a partition of the support of ϕ for all $\ell \in \{0\} \cup \mathbb{N}$. Then, $\overline{\phi}_\ell$ can be computed as a distributed sum of contributions $\overline{\phi}_\ell(q) = \sum_{i=1}^n \overline{\phi}_\ell^i(q)$, for all $q \in Q$, if agents can communicate with neighbors in the (connected) graph $\mathcal{G}_{\mathrm{NN}}(\ell)$, $\ell \in \overline{\mathbb{Z}_{\geq 0}}$, when using a NN interpolation scheme or, respectively, in the graph $\mathcal{G}_{\mathrm{LIDW}}(\ell)$ when using a LIDW interpolation scheme. Here $\overline{\phi}_\ell^i(q) = f_\ell^i(q) \cdot 1_{D_i^\ell}(q)$ is maintained by each vehicle, where:

$$f_\ell^i(q) = \sum_{j\in\mathcal{N}_i^{\mathrm{NN}}(\ell)} f_{\ell-1}^j(q) 1_{D_j^{\ell-1}}(q) + W_\ell^{\mathrm{NN}}(\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell)) \,,$$

$$f^i_\ell(q) = \sum_{j \in \mathcal{N}_i^{\mathrm{LIDW}}(\ell)} f^j_{\ell-1}(q) 1_{D_j^{\ell-1}}(q) \qquad (8)$$
$$+ W_\ell^{\mathrm{LIDW}} w_i^\ell(q)(\phi(p_i^\ell) - \overline{\phi}_{\ell-1}(p_i^\ell)) \,, \quad \text{in the LIDW case}\,,$$

$\overline{\phi}_{\ell-1}(p_j^\ell) = f^{k_j}_{\ell-1}(p_j^\ell)$ with $p_j^\ell \in D_{k_j}^{\ell-1}$, for all $\ell \in \mathbb{N}$, and $f_0^j(q) = \phi(p_j^0)$, in the NN case, resp. $f_0^j(q) = \phi(p_j^0) w_j^0(q)$, in the LIDW case, for all $j \in \{1, \dots, n\}$.

Proof: We will prove the result by induction. We will provide computations for the NN and LIDW cases in parallel. Let $\ell = 1$. By definition, see (1),

$$\overline{\phi}_1 = \overline{\phi}_0 + W_1^*(\overline{\mathcal{Q}}_1 \phi - \overline{\mathcal{Q}}_1 \overline{\phi}_0) \,. \qquad (9)$$

Since $\{D_j^\ell\}_{j=1}^n$ is a partition of the support of ϕ for all $\ell \in \mathbb{N}$:

$$\overline{\phi}_0(q) = \sum_{k=1}^n \phi(p_k^0) 1_{D_k^0}(q) = \sum_{j=1}^n \big(\sum_{k=1}^n \phi(p_k^0) 1_{D_k^0}(q)\big) 1_{D_j^1}(q) \,.$$

Since $1_{D_k^0}(q) 1_{D_j^1}(q) \neq 0$ iff $k \in \mathcal{N}_j^{\mathrm{NN}}(1)$ (resp. $k \in \mathcal{N}_i^{\mathrm{LIDW}}(1)$), then we have:

$$\overline{\phi}_0(q) = \sum_{j=1}^n \big(\sum_{k \in \mathcal{N}_j^{\mathrm{NN}}(1)} \phi(p_k^0) 1_{D_k^0}(q)\big) 1_{D_j^1}(q) \,, \text{ for NN},$$
$$\overline{\phi}_0(q) = \sum_{j=1}^n \big(\sum_{k \in \mathcal{N}_j^{\mathrm{LIDW}}(1)} \phi(p_k^0) w_k^0(q) 1_{D_k^0}(q)\big) 1_{D_j^1}(q) \,, \text{ for LIDW}.$$

Using this fact and equation (9), we obtain:

$$\overline{\phi}_1(q) = \sum_{j=1}^n \big(\sum_{k \in \mathcal{N}_j^{\mathrm{NN}}(1)} \phi(p_k^0) 1_{D_k^0}(q) + W_1^{\mathrm{NN}}(\phi(p_j^1) - \overline{\phi}_0(p_j^1))\big) \times$$
$$\times 1_{D_j^1(q)} = \sum_{j=1}^n f_0^j(q) 1_{D_j^1(q)} \,, \quad \text{for NN},$$
$$\overline{\phi}_1(q) = \sum_{j=1}^n \big(\sum_{k \in \mathcal{N}_j^{\mathrm{LIDW}}(1)} \phi(p_k^0) w_k^0(q) 1_{D_k^0}(q)$$
$$+ W_1^{\mathrm{LIDW}}(\phi(p_j^1) - \overline{\phi}_0(p_j^1))\big) \times 1_{D_j^1}(q)$$
$$= \sum_{j=1}^n f_0^j(q) 1_{D_j^1}(q) \,, \quad \text{for LIDW}.$$

Let $f_{\ell-1}^j(q)$, $j \in \{1, \dots, n\}$, $\ell \geq 2$, be defined as in (8) and suppose that $\overline{\phi}_{\ell-1}(q) = \sum_{i=1}^n f^j_{\ell-1}(q) 1_{D_i^{\ell-1}}(q)$. Using that $\{D_j^\ell\}_{j=1}^n$ is a partition of the support of ϕ for all $\ell \in \mathbb{N}$, then:

$$\overline{\phi}_{\ell-1}(q) = \sum_{i=1}^n f^i_{\ell-1}(q) 1_{D_i^{\ell-1}}(q)$$
$$= \sum_{k=1}^n \big(\sum_{i=1}^n f^i_{\ell-1}(q) 1_{D_i^{\ell-1}}(q)\big) 1_{D_k^\ell}(q)$$
$$= \sum_{k=1}^n \big(\sum_{i \in \mathcal{N}_k^*(\ell)} f^i_{\ell-1}(q) 1_{D_i^{\ell-1}}(q)\big) 1_{D_k^\ell}(q) \,.$$

Further, by equation (1), we obtain the following computations that prove the induction:

$$\overline{\phi}_\ell = \sum_{k=1}^n \big(\sum_{i \in \mathcal{N}_k^{\mathrm{NN}}(\ell)} f^i_{\ell-1}(q) 1_{D_i^{\ell-1}}(q)$$
$$+ W_\ell^{\mathrm{NN}}(\phi(p_k^\ell) - \overline{\phi}(p_k^\ell))\big) 1_{D_k^\ell}(q) \,, \quad \text{for NN}\,,$$
$$\overline{\phi}_\ell = \sum_{k=1}^n \big(\sum_{i \in \mathcal{N}_k^{\mathrm{LIDW}}(\ell)} f^i_{\ell-1}(q) 1_{D_i^{\ell-1}}(q)$$
$$+ W_\ell^{\mathrm{LIDW}} w_k^\ell(q)(\phi(p_k^\ell) - \overline{\phi}(p_k^\ell))\big) 1_{D_k^\ell}(q) \,, \text{ for LIDW}\,.$$

■

Remark 7 (Vehicle motions and network density): From here we see that, in order to maintain a data-base representation of ϕ in the current dominance region, each vehicle needs to communicate with others that were contributing to the estimation over portions of its region one time-step before. An assumption is that $\{D_i^\ell\}_{i=1}^n$ is a partition of the support of ϕ. This is always true for a Voronoi partition, but might not hold for balls $B(p_i, R)$ with small $R > 0$ unless there is a sufficiently large number of vehicles (a dense network). In case the number of vehicles is not large enough, the scheme only serves to update the function over a limited area. On the other hand, the motion of the vehicles may also be restricted in order to guarantee communication with others at the expense of getting less efficient estimates of the gain W. For example, in the limited range case, where the sensing radius is R (equal to the correlation radius of the LIDW) and the communication radius is $2R$, it can be guaranteed that vehicles move to regions of agents which are within communication range by restricting their motion to a range of $R/2$. In this way, the vehicles would have to move toward their approximated centroids as much as possible but stay within $R/2$. •

Remark 8 (Information loss): The question of information or measurement loss is an important issue that we have consciously left out of the paper. A thorough treatment; e.g. as in [20], also connects this problem with asynchronocity, which we mention as future work. However, assuming that there is a procedure to acknowledge measurement receipt from neighbors, a version of the filter that only considers measurements that were successfully transmitted can be implemented. In this way, the computation of the global gain W will take into account only those successfully transmitted measurements to update the filter. Since other than this, the computation of W is robust to other communication failures (as explained in Theorem 5), the algorithm can then proceed. The information loss will mainly affect the covariance of the error of the filter and the limit value of the MMSE. In particular, since ϕ is static, the filter will still make the covariance and the MMSE approach a bounded limit. However, the value of this limit and the velocity of convergence to it will be affected. •

Remark 9 (Data Compression): As more measurements are gathered, the load of information to be transmitted between vehicles increases. In order to maintain the overall process scalable, some sort of a data compression procedure should be implemented. Any compression procedure will induce a modification of the filter as we describe in the following.

Apart from the gain information, the data that each agent needs to transmit to others is basically encoded in $\overline{\phi}^i_\ell(q)$, $i \in \{1,\dots,n\}$. Let $\mathcal{C}_i : \mathscr{C} \to \mathscr{C}$ be a compression method for agent i. This induces the general expression $\mathcal{C}\overline{\phi}_\ell(q) = \sum_{i=1}^n \mathcal{C}_i \overline{\phi}^i_\ell(q)$, which, in turn, changes the filter update (1) according to:

$$\overline{\phi}_\ell(q) = \mathcal{C}\overline{\phi}_{\ell-1}(q) + \mathcal{W}_\ell(\overline{\mathcal{Q}}_\ell\phi(q) - \overline{\mathcal{Q}}_\ell\mathcal{C}\overline{\phi}_{\ell-1}(q))\,.$$

In this way, the expression of the optimal gain should also change according to:

$$\mathcal{W}^*_\ell = \frac{\int_Q(\phi - \mathcal{C}\overline{\phi}_{\ell-1})(\overline{\mathcal{Q}}_\ell\phi - \overline{\mathcal{Q}}_\ell\mathcal{C}\overline{\phi}_{\ell-1})dq + C}{\int_Q(\overline{\mathcal{Q}}_\ell\phi - \overline{\mathcal{Q}}_\ell\mathcal{C}\overline{\phi}_{\ell-1})^2 dq + C}\,,$$

see the proof of Theorem 5.

Particular compression procedures that vehicles can implement can also be based on interpolations such as a Lloyd quantization method as explained next. In general, each $\overline{\phi}^i_\ell(q)$ can be expressed as a sum:

$$\overline{\phi}^i_\ell(q) = f^i_\ell(q)1_{D^\ell_i}(q) = \sum_{k=1}^M a^\ell_{ik}(q)1_{A^\ell_{ik}}(q)\,, \qquad (10)$$

for some sets $A^\ell_{ik} \subseteq D^\ell_i$ with disjoint interiors, and $a^\ell_{ik}(q) \in \mathbb{R}$ which are linear combination of measurements weighted by $w^r_j(q)$ functions. A possible way to compress (10) is to limit the number of summands to $m \ll M$. That is, take the approximation

$$\sum_{k=1}^M a^\ell_{ik}(q)\,1_{A^\ell_{ik}}(q) \approx \sum_{k=1}^m b^\ell_{ik}\,1_{B^\ell_{ik}}(q)\,,$$

using a lower dimensional set of $b^\ell_{ik} \in \mathbb{R}$ and $B^\ell_{ik} \subseteq D^\ell_i$, $k \leq m$. To do this, a Lloyd quantization algorithm can be implemented by each vehicle as follows. First, m initial conditions are chosen, $\mathcal{F}(0) = \{q_k \in D^\ell_i \,|\, k \in \{1,\dots,m\}\}$, $b^\ell_{ik}(0) = \overline{\phi}^i_\ell(q_k)$ and $B^\ell_{ik}(0) = V_k(\mathcal{F}(0))$, $k \in \{1,\dots,m\}$. Then, these are updated for a number of times according to $\mathcal{F}(s+1) = \{C_{B^\ell_{ik}(s)} \,|\, C_{B^\ell_{ik}(s)}$ is the centroid of $B^\ell_{ik}(s)\}$, $b^\ell_{ik}(s+1) = \overline{\phi}^i_\ell(C_{B^\ell_{ik}(s)})$, $B^\ell_{ik}(s+1) = V_k(\mathcal{F}(s))$, $k \in \{1,\dots,m\}$ and $s \geq 1$. The iteration leads to a suboptimal approximation of (10) using m summands. •

Remark 10 (Relation to coverage problems): The prescription of the motion of the vehicles to the approximated centroids of their regions corresponds to what is done in coverage algorithms proposed in [21]. Clearly, as the approximations $\overline{\phi}_\ell(q)$ become more and more accurate, the approximated centroids will become as well. In this way, the final configuration of the vehicles will converge to a *centroidal Voronoi* configuration, where agents are placed at the centroids of their dominance regions. This configurations are suboptimal with respect to a multi-center function measuring coverage. •

V. Simulations

This section presents two simulation experiments of the proposed NN and LIDW filters for the estimation of $\phi(q) = 0.05 + 3\exp^{-\frac{1}{2}\|q-r_1\|^2} + 3\exp^{-\frac{1}{2}\|q-r_2\|^2} + 3\exp^{\|q-r_1\|^2}$, with $r_1 = (8,2)$, $r_2 = (8,4)$ and $r_3 = (3,7)$ over a square area of 10×10 area units. The variance of the white noise in the measurement model is taken to be 0.05. Figure 1 presents the estimate of ϕ through the NN interpolation filter using 75 sensors with a limited range of 0.5. Figure 2 shows the estimate of ϕ through LIDW using 64 sensors. The number of time steps used in the computation of the gain is $T = 25$. These simulations do not consider loss of resolution. The smaller number of sensors employed for the LIDW makes the MMSE a bit larger. We have implemented the filter with a time-varying ϕ moving on a circle. Starting with absolute knowledge of the density, one can see the MMSE reach a similar bound with these filters.

VI. Conclusions

We have presented adaptive interpolation schemes for slowly time-varying field estimation. Conditions for the possible decentralization of the schemes were obtained and require (1) mild connectivity assumptions for the computation of the gains, and (2) communication between neighbors in the graphs $\mathcal{G}_{\mathrm{NN}}(\ell)$ and $\mathcal{G}_{\mathrm{LIDW}}(\ell)$. Future work will be dedicated to the investigation of distributed schemes using global interpolations and kriging methods, as well as the effect of measurement loss in the filter performance.

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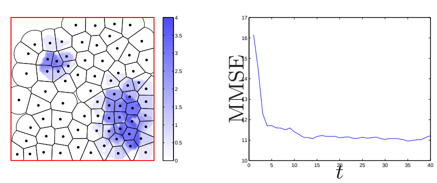

Fig. 1. An implementation of the NN decentralized interpolation filter.

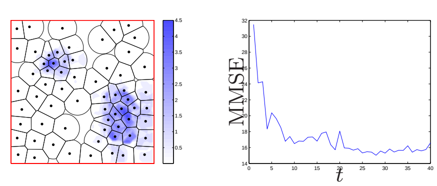

Fig. 2. An implementation of the NN decentralized interpolation filter.

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