Self-triggered Best-Response Dynamics for Continuous Games

Andres Cortés and Sonia Martínez

Abstract

Motivated by the development of novel and practically implementable coordination algorithms for multi-agent systems, here we investigate the adaptation of classical best-response dynamics by means of self-triggered communications. First, we show that, if the best-response dynamics is defined for a continuous-action-space potential game, convergence towards the Nash Equilibria set is guaranteed under some continuity assumptions on utilities and component-wise concavity on the potential function. Then, we modify the best-response dynamics to account for a self-triggered communication strategy, with the aim of producing economic communications while ensuring convergence to the equilibrium set. The proposed algorithm is then analyzed using hybrid systems theory. We illustrate the results in an example of autonomous agents for their optimal deployment on a one-dimensional environment. Finally, we present some simulations that demonstrate the performance of the proposed strategy for the sensor network.

I. Introduction

The last years have witnessed an intense research activity in the development of novel distributed algorithms for multi-agent systems with performance guarantees. A particular effort has been devoted to the study of game-theoretic approaches that can model and regulate selfish agent interactions. By means of these, the multi-agent coordination objective is formulated in terms of Nash Equilibria (NE), which correspond to the natural emergent behavior arising from the interaction of selfish players. Due to their modularity, game dynamics can easily be implemented by agents relying on local information, leading to a robust performance. Even though the resulting emerging behavior may not be optimal, it is generally expected that the behavior is as close as possible to that of the benchmark given by a centralized design. Modularity also leads to a more robust performance when facing local failures. However, finding algorithms to reach a NE is not always an easy task, mainly due to the fact that for some games, the NE is very difficult to compute, and even some games do not have any.

The best-response dynamics describes an interaction in which each player is able to compute its own best action against other players' action profile. Then, the player's action evolves continuously towards its best-response set. Convergence of the best-response dynamics has been studied for games under well defined conditions. In [1], this convergence is proven for finite zero-sum games with bilinear payoff functions. In [2] the authors study convergence of the best-response dynamics for potential games with continuously differentiable potential function. Unlike these results, our work considers potential games for which the potential function is not differentiable everywhere. In [3], the authors consider best response dynamics for two-player zero-sum games, with concave and convex payoff functions. Convergence to the saddle point set is proven, since this set corresponds to the NE set of the game. In [4], the authors extend the above result to a two-player zero-sum continuous game with quasiconcave and quasiconvex continuous payoff functions. Our work differs from these because we consider n-player potential games with component-wise

A. Cortés and S. Martínez are with Department of Mechanical and Aerospace Engineering, University of California, San Diego,9500 Gilman Drive La Jolla, CA 92093-0411{aicortes@ucsd.edu, soniamd@soe.ucsd.edu}.

pseudo-concave, component-wise quasiconcave, potential function. For discrete-time dynamics (iterated games) in potential games with bilinear payoffs, a NE can be reached [5].

Probably the most favored field with developments in multi-agent control, is the control of robotic networks, which includes coverage control. The objective is to allocate a set of agents, each one with a specific radius of action, to a bounded environment, in such a way that the final agents' position maximize certain performance metric [6]. A recent body of work formulates coverage control by means of non-cooperative games. In this way, each (selfish) player tries to maximize its own utility (sensing quality) in a competitive way leading eventually to an emergent global welfare (area coverage). Works in [7], [8] formulate sensor network deployment as potential games, in a finite action environment. These papers adapt learning mechanisms from Game Theory to handle partial information and constrained motions in sensor networks.

The idea of restricting communication efforts to time instants at which it is absolutely necessary to have current information leads to the self-triggered and event-triggered concepts; e.g. see [9], [10], [11] and references therein. This idea has been recently extended to the context of multi-agent systems and distributed optimization with applications to cooperative control [12], [13], [14], [15]. The present work contributes further to this area by studying a complementary game-theoretic setting. A main difference with gradient-based methods is given by the need of estimating the evolution of best-response sets as agents' actions change with time.

More precisely, we start by analyzing the convergence properties of the continuous-time best-response dynamics for a continuous-action-space game by means of the invariance theory for set-valued dynamical systems. We show that all the solutions of the best-response dynamics converge to the NE set of the potential game for component-wise pseudo-concave, component-wise quasiconcave potential functions. The continuous-time best-response dynamics is not practically implementable as it requires continuous communications, thus we introduce a novel self-triggered best-response dynamics relying on Lipschitz payoff functions. Then, we prove how this strategy still ensures convergence to the set of NE while regulating the communication effort according to its need. The results are applied to a 1D agent deployment problem formulated as a potential game, where agents can vary both their position and their coverage radius. A similar result has been presented in [16], where the self-triggered best-response dynamics is used for 1D agent deployment where agents can vary their positions, but have fixed radius. A self-triggering law based on the Lipschitz c ontinuity of the best-response sets for all players is introduced.

The paper is organized as follows. In Section II, we introduce briefly some game theoretical concepts that are used throughout the paper. Section III contains the description of the best-response dynamics, as well as its convergence analysis towards the NE set. In Section IV, we present our self-triggered communication strategy and show convergence of the solutions to the NE set. Then, in Section VI, we present a complete agent-deployment example modeled as a potential game that holds the properties to use the best-response dynamics. We show some simulations demonstrating the performance of the proposed self-triggered best-response dynamics on the example.

Notations. In what follows, sign : $\mathbb{R} \to \mathbb{R}$ is defined as $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = -1$ if $x < 0$, and $\text{sign}(x) = 0$ if $x = 0$. Let S be a subset of $\mathbb{R}^n$, then $\text{co}(S)$ denotes the convex hull of S, $\overline{S}$ denotes the closure of S, and $\rho(S)$ denotes the diameter of S, $\rho(S) = \sup_{x,y\in S} \|x - y\|$. If f is a map, $\text{dom}(f)$ represents the domain of f. The open ball with radius r centered at x is denoted as $B_r(x)$. Given $f : \mathbb{R}^n \to \mathbb{R}$, we define a level set of f as $f^{-1}(r) = \{x \in \text{dom}(f) \mid f(x) = r\}$. For A, B two subsets of $\mathbb{R}^n$, we denote $A \setminus B = \{x \in A \mid x \notin B\}$. Let A be a subset of $\mathbb{R}^n$. Then, $\text{int}(A)$ represents the interior, and $\text{bnd}(A)$ represents the boundary of the set.

II. Game Theoretical Notions

In this section, we first introduce some basic definitions from Game Theory [5] and an adaptation from [8] to deal with constrained motion coordination problems.

***Definition** 2.1:* A *continuous-action-space game* is a 3-tuple $\Gamma = (I, X, u)$, such that (i) $I = \{1, \ldots, N\}$ is the set of N players, (ii) $X = \prod_{i=1}^{N} X_i \subset \mathbb{R}^d$ is the action space of the game, with $X_i \subset \mathbb{R}^{n_i}$, $i \in I$, $d = \sum_i n_i$, a compact and convex set representing the action space of the i^{th} player, and (iii) $u : X \to \mathbb{R}^N$ is a function whose component $u_i : X \to \mathbb{R}$ defines the payoff of the i^{th} player, $i \in I$.

As opposed to discrete-action games, note that the possible action of a player can take values in a compact and convex set, resulting into a continuous-action-space game.

Let $x_i \in X_i \subset \mathbb{R}^{n_i}$ be the action for the i^{th} player and $x \in X$ be the action profile for all players, such that $x = (x_1, \ldots, x_N)^\top$. In the sequel, we will use the notation $x = (x_i, x_{-i})$, where $x_{-i} \in X_{-i} = \prod_{j \in I, j \neq i} X_j$, for all $i \in I$, are the actions of all players except that of the i^{th} player.

A repeated, continuous-time, game associated with Γ, $\mathcal{R}(\Gamma)$, is a game in which, at each time $t \in \mathbb{R}_{\geq 0}$, each agent $i \in I$ modifies $x_i(t) \in X_i$ simultaneously while receiving $u_i(x(t))$. This is in contrast to repeated, discrete-time games, which follow a discrete-time schedule.

In the context of (vehicle) motion coordination, agents' actions can be identified with system states, and thus it makes sense that these change in continuous time according to some vehicle dynamics. In particular, the way in which player i modifies $x_i(t)$ can be constrained by a (state-dependent) set W. Let $W_i(x_i, x_{-i}) \subset X_i$ be a constraint subset associated with $x \in X$, $W(x) = \Pi_{i \in I} W_i(x) \subset X$, and $W = \cup\{(x, W(x)) \mid x \in X\} \subseteq X \times X$. We will refer to W as a *fiber bundle* over X. The introduction of W leads to the notion of constrained repeated game associated with Γ and W, $\mathcal{R}_W(\Gamma)$, and the following equilibrium concept.

***Definition** 2.2:* Let $\Gamma = (I, X, u)$ be a continuous-space game and W a fiber bundle over X. A *constrained Nash Equilibrium (NE)* for Γ with respect to W is an action profile $(x_i^\star, x_{-i}^\star) \in X$ such that $u_i(x_i, x_{-i}^\star) \leq u_i(x_i^\star, x_{-i}^\star)$, for all $x_i \in W_i(x_i^\star, x_{-i}^\star)$ and all $i \in I$.

We will use such W to represent collision-avoidance type of constraints, or restricted reachable sets, thus it will be additionally assumed that $x \in W(x)$. For example, a velocity saturation constraint of $v_{\max}$ may simply be expressed by the limited reachable set $W_i(x) = W_i(x_i) = \overline{B}_{v_{\max}}(x_i)$.

Out of different classes of games, the notion of potential game [17] is of particular interest since the incentives for all players can be captured by a single function.

***Definition** 2.3:* Consider a game $\Gamma = (I, X, u)$. Let us assume that there exists a function $\Phi : X \to \mathbb{R}$ such that

$$\operatorname{sign}\left(u_i(x_i, x_{-i}) - u_i(x_i', x_{-i})\right) = \operatorname{sign}\left(\Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i})\right),$$

for $x_i, x_i' \in X_i$, $x_{-i} \in X_{-i}$, for all $i \in I$. Then, the game is called an *ordinal potential game.* Moreover, if

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = \Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i}),$$

for $x_i, x_i' \in X_i$, $x_{-i} \in X_{-i}$, then the game is called an *exact potential game.*

III. Continuous-time best-response dynamics

Here, we introduce some basic facts about continuous-time best-response dynamics [18] and show their convergence to the set of equilibria under some general conditions.

Let $\Gamma = (I, X, u)$ be a continuous-space game, let W be a continuous fiber bundle over X, such that $W_i(x)$ convex and compact for all $x \in X$, $i \in I$, and consider the constrained repeated game associated with Γ and W, $\mathcal{R}_W(\Gamma)$.

***Definition** 3.1:* The *best-response dynamics* for $\mathcal{R}_W(\Gamma)$ is defined by the differential inclusion $F : X \rightrightarrows \mathbb{R}^d$, $F_i(x) = \mathrm{BR}_i(x_{-i}) - x_i = \mathrm{argmax}_{y \in W_i(x)} u_i(y, x_{-i}) - x_i$, for all $i \in I$. That is,

$$\dot{x}_i \in F_i(x) := \mathrm{BR}_i(x_{-i}) - x_i, \quad i \in I. \tag{1}$$

We denote $F(x) = \mathrm{BR}(x) - x$ for conciseness. Existence of solutions for differential inclusions is guaranteed for F, nonempty, upper semicontinuous and taking compact and convex values. Let us assume that payoff functions u_i are continuous maps on X. By compactness of X, u_i reaches its maximum value on X, and the set of maximizers is compact. Then, F_i is nonempty and takes compact values. Further, let us assume that u_i is quasiconcave on W. Then, the set of maximizers of u_i and F_i are convex for each $x \in W$[1]. By continuity of u_i on X, and continuity of W, we can apply directly the maximum theorem [19] to conclude that F_i is upper semicontinuous for each $i \in I$. Alternatively, in potential games, one can exchange the continuity assumption on the u_i by continuity on Φ. Since F_i is nonempty, compact, convex and upper semicontinuous at every $x \in X$, and each $i \in I$, there exists a solution to (1) for every initial condition. These solutions are absolutely continuous functions, $\varphi : [0, +\infty) \to X$, such that $\dot{\varphi}_i(t) \in \mathrm{BR}_i(\varphi_{-i}(t)) - \varphi_i(t)$, for almost every $t \in [0, +\infty)$, and for all $i \in I$; see [20]. The equilibria set of system (1) is

$$X^\star = \{x \in X \mid x_i \in \mathrm{BR}_i(x_{-i}),\ \forall i \in \{1, \ldots, N\}\}. \tag{2}$$

This set corresponds exactly to the set of constrained Nash equilibria for Γ with respect to W.

The above theorem will be used to analyze the best-response dynamics associated with a potential game. The potential function should satisfy the following property.

***Definition** 3.2:* Let $Y \subseteq \mathbb{R}^d$ be a convex set. A potential function $\Phi : Y \to \mathbb{R}$ is said to be *component-wise pseudo-concave* (respectively *component-wise pseudo-convex*) if for every $i \in I$, and every $w = (w_i, w_{-i}), y = (y_i, y_{-i}) \in Y$, and $s \in (0, 1)$, with $y_{-i} = w_{-i}$, it holds that if $\Phi(w) > \Phi(y)$, then $\Phi(sw_i + (1-s)y_i, w_{-i}) \geq s\Phi(y_i, w_{-i}) + (1-s)sb(w_i, y_i)$ where $b(w_i, y_i)$ is a positive function (respectively if $\Phi(w) < \Phi(y)$, then $\Phi(sw_i + (1-s)y_i, w_{-i}) \leq \Phi(y_i, w_{-i}) + (1-s)sb(w_i, y_i)$ where $b(w_i, y_i)$ is a negative function).

Similarly, Φ is said to be *component-wise quasiconcave* (respectively *component-wise quasiconvex*) if for every $i \in I$, and every $w = (w_i, w_{-i}), y = (y_i, y_{-i}) \in Y$, and $s \in (0, 1)$, with $y_{-i} = w_{-i}$, it holds that $\Phi(sw_i + (1-s)y_i, w_{-i}) \geq \min\{\Phi(w_i, w_{-i}), \Phi(y_i, w_{-i})\}$ (respectively if $\Phi(sw_i + (1-s)y_i, w_{-i}) \leq \max\{\Phi(w_i, w_{-i}), \Phi(y_i, w_{-i}))\}$).

In the following, we make use of Theorem 1.1 to show convergence of the best-response dynamics to its equilibria set under certain conditions. Definitions of regular function, generalized gradient, set-valued Lie derivative, and generalized LaSalle's Invariance Principle can be found in Appendix A, B.

***Theorem** 3.1:* Let $\Gamma = (I, X, u)$ be an ordinal potential game with potential function Φ. Let W be a continuous fiber bundle over X such that $W_i(x)$ is compact and convex for all $x \in X$, $i \in I$. Assume that Φ is component-wise

[1] Alternatively, the same result holds for a potential game with a component-wise quasiconcave potential function (Definition 3.2).

quasiconcave, component-wise pseudoconcave with b_i continuous over each $W(x)$, $x \in X$, Lipschitz, and regular over X. Let $F : X \rightrightarrows \mathbb{R}^d$ be the best-response dynamics for $\mathcal{R}_W(\Gamma)$. Then, all solutions of the system $\dot{x} \in F(x)$ converge to the set $X^\star$ of constrained Nash equilibria defined in (2).

Proof: Consider $\Psi = -\Phi$. Since Φ is component-wise pseudoconcave, then Ψ is component-wise pseudoconvex. We will see that Ψ is a Lyapunov function for our set-valued map; that is, it holds that $\max \mathcal{L}_F \Psi(x) \leq 0$, for all $x \in X$.

Let x be a point in X. Any $v \in F(x)$ has the form $v = x^\star - x$, with $x^\star \in \text{BR}(x)$. Define $\Omega_\Psi \subset X$ as the zero-measure set for which Ψ is non-differentiable. Consider a $\zeta \in \partial\Psi(x)$ of the form $\zeta = \lim_k \nabla\Psi(y^k)$, with $y^k \to x$, $y^k \notin \Omega_\Psi$. If $x^\star = x$, then it trivially holds that $v^T\zeta = 0$. Suppose that $x^\star \neq x$. Since BR is nonempty and upper-semicontinuous for all $x \in X$, it holds that there exists a sequence $x^{k,\star} \to x^\star$ such that $x^{k,\star} \in \text{BR}(y^k)$, for all k. Thus, we have $v^T\zeta = (x^\star - x)^T \lim_k \nabla\Psi(y^k) = \lim_k (x^{k,\star} - y^k)^T \nabla\Psi(y^k)$.

Let us define $\nabla_i\Psi \in \mathbb{R}^{n_i}$ as the partial derivative of Ψ with respect to the action of the i^{th} player. Since Ψ is differentiable at y^k, the term $(x^{k,\star} - y^k)^T \nabla\Psi(y^k)$ is the directional derivative of Ψ at y^k along the direction $x^{k,\star} - y^k$. In particular,

$$\begin{aligned}
v^T\zeta &= \lim_{k\to\infty} (x^{k,\star} - y^k)^T \nabla\Psi(y^k) \\
&= \lim_{k\to\infty} \sum_{i\in I} (x_i^{k,\star} - y_i^k) \nabla_i \Psi(y^k) \\
&= \lim_{k\to\infty} \sum_{i\in I} (x_i^{k,\star} - y_i^k, 0_{-i})^T \nabla\Psi(y^k) \\
&= \lim_{k\to\infty} \sum_{i\in I} \lim_{h\to 0} \frac{\Psi(y_i^k + h(x_i^{k,\star} - y_i^k), y_{-i}^k) - \Psi(y^k)}{h} ,
\end{aligned}$$

where in the last equality we have used the limit definition of directional derivative.

Notice that since $x^{k,\star} \in \text{BR}(y^k)$, then it holds that $\Psi(y^k) \geq \Psi(x_i^{k,\star}, y_{-i}^k)$ for any $i \in I$. Moreover, since $x \neq x^\star$, we have that there is a $k_1 < \infty$ for which $y^k \neq x^{k,\star}$ for all $k > k_1$. Next, assume that $x \notin \text{BR}(x)$, then there is an $i \in I$ such that $x_i \notin \text{BR}_i(x)$. By continuity of Ψ, the set $\text{BR}(x)$ is closed, therefore for each $x_i \in W_i(x) \setminus \text{BR}_i(x)$ there exists ε such that $B_\varepsilon(x) \cap W_i(x) \subset W_i(x) \setminus \text{BR}_i(x)$. Therefore, there is $k_2 < \infty$ such that $y_i^k \notin \text{BR}_i(x^{k,\star})$ for all $k > k_2$. Thus, when we study the behavior as $k \to \infty$, we will consider only sequences y^k such that $y^k \neq x^{k,\star}$ and $y^k \notin \text{BR}(y^k)$.

Using the fact that $y_i^k \notin \text{BR}_i(y^k)$ and by component-wise pseudoconvexity of Ψ, it holds that since $\Psi(x_i^{k,\star}, y_{-i}^k) < \Psi(y^k)$, then $\Psi(y_i^k + h(x_i^{k,\star} - y_i^k), y_{-i}^k) \leq \Psi(y_i, y_{-i}^k) + (1-h)hb_i(x_i^{k,\star}, y_i^k)$, for any $h \in (0,1)$, and each $i \in I$. From here, $\Psi(y_i^k + h(x_i^{k,\star} - y_{-i}^k), y_{-i}^k) - \Psi(y^k) \leq (1-h)hb_i(x_i^{k,\star}, y_i^k)$, which implies that $\lim_{h\to 0} \frac{\Psi(y_i^k + h(x_i^{k,\star} - y_i^k), y_{-i}^k) - \Psi(y^k)}{h} \leq b_i(x_i^{k,\star}, y_i^k)$. Now, for each $j \in I$ such that $y_j^k \in \text{BR}_j(y^k)$, we have that $\Psi(y^k) = \Psi(x_j^{k,\star}, y_{-j}^k)$. It means that y_j^k and $x_j^\star$ are minimizers of $\Psi(\cdot, y_{-j}^k)$. By component-wise quasiconvexity, the set of minimizers of $\Psi(\cdot, y_{-j}^k)$ is convex, then $\Psi(y_j^k + h(x_j^{k,\star} - y_j^k), y_{-j}^k) = \Psi(y^k) = \Psi(x^{k,\star})$, therefore we can conclude that $\lim_{h\to 0} \frac{\Psi(y_i^k + h(x_i^{k,\star} - y_i^k), y_{-i}^k) - \Psi(y^k)}{h} = 0$.

Then, it follows by using the continuity of b_i that

$$\begin{aligned} v^T\zeta &= \lim_{k\to\infty}(x^{k,\star}-y^k)\nabla\Psi(y_k) \\ &\le \lim_{k\to\infty} \sum_{\substack{i\in I \\ y_i^k\notin \mathrm{BR}_i(y^k)}} b_i(x_i^{k,\star}, y_i^k) = \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(x_i^\star, x_i). \end{aligned} \tag{3}$$

Now, consider the case when $x \in \mathrm{BR}(x)$. In this case, if $x \in \mathrm{int}(\mathrm{BR}(x))$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathrm{BR}(x)$. Note that for $h \to 0$, $(x_i + h(x_i^\star - x_i), x_{-i}) \in B_\varepsilon(x) \subset \mathrm{BR}(x)$. Then, $\Psi(x_i + h(x_i^\star - x_i), x_{-i}) = \Psi(x_i, x_{-i})$, and $\lim_{h\to 0} \frac{\Psi(x_i+h(x_i^\star-x_i),x_{-i})-\Psi(x)}{h} = 0$, for each $i \in I$. It implies that $v^T\zeta = 0$. If $x \in \mathrm{bnd}(\mathrm{BR}(x))$, there are: i) sequences $y^k \to x$ such that for every k, $y^k \notin \mathrm{BR}(y^k)$, ii) sequences such that $y^k \in \mathrm{BR}(y^k)$ for all k, and iii) sequences such that there is a subsequence $\{y^{k,l}\}_l \subset \mathrm{BR}(y^k)$ and a subsequence $\{y^{k,l}\}_l \subset W(x) \setminus \mathrm{BR}(y^k)$. In case i) we follow the same analysis as that for $x \notin \mathrm{BR}(x)$, then $v^T\zeta < 0$, in case ii) the analysis is analogous to that for $x \in \mathrm{int}(\mathrm{BR}(x))$, to show that $v^T\zeta = 0$. In the third case, if x is a point of non-differentiability, the gradient of Ψ at y^k does not converge, then we do not need to consider these sequences. If Ψ is differentiable at x, then, as x_i is a minimizer of $\Psi(\cdot, x_{-i})$, for all $i \in I$, we have that $v^T\zeta = 0$. Hence, $v^T\zeta \le 0$ for all $x \in \mathrm{bnd}(\mathrm{BR}(x))$. Then, we can conclude that for all $x \in X$, it holds that $v^T\zeta \le 0$ for all sequences $y^k \to x$, such that $y^k \notin \Omega_\Psi$ and $\lim_k \nabla\Psi(y^k) = \zeta$.

Now consider any $\zeta \in \partial\Psi(x)$. By the definition of *generalized gradient*, there exist $\alpha_1, \dots, \alpha_l$, with $0 \le \alpha_s \le 1$, and $\alpha_1 + \cdots + \alpha_l = 1$, and sequences $\{y^{k_1}\}, \dots, \{y^{k_l}\}$ converging to x such that $\zeta = \alpha_1 \lim_{k_1} \nabla\Psi(y^{k_1}) + \cdots + \alpha_l \lim_{k_l} \nabla\Psi(y^{k_l})$. Then it follows that $v^T\zeta = \alpha_1 v^T\zeta_1 + \cdots + \alpha_l v^T\zeta_l$. Using the previous analysis for each ζ_s, it follows that $v^T\zeta \le 0$. From here we conclude that $\max \mathcal{L}_F\Psi(x) \le 0$ for all $x \in X$. From the generalized LaSalle's invariance principle, we have that all solutions will converge to the largest invariant set contained in $X \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \mathcal{L}_F\Psi\}}$. In the following, we prove that the largest invariant set is contained in $X^\star$.

Suppose that $x \notin \mathrm{BR}(x)$, and x belongs to the invariant set. Take a $x^\star \in \mathrm{BR}(x)$, define $v = x^\star - x$, and take a $\zeta \in \partial\Psi(x)$ such that $\zeta = \lim_k \nabla\Psi(y^k)$, with $y^k \to x$, when $k \to +\infty$. From (3), we have that $v^T\zeta \le \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(x_i^{k,\star}, y_i^k) < 0$, where, the second inequality follows from the fact that there is a $j \in I$ such that $x_j \notin \mathrm{BR}_j(x_{-j})$. Taking the maximum over $\mathrm{BR}(x)$, we have that

$$\max_{x^\star\in \mathrm{BR}(x)} (x^\star - x)^T\zeta = \max_v v^T\zeta \le \max_{x^\star\in\mathrm{BR}(x)} \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(x_i^\star, x_i) = \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(\bar{x}_i^\star, x_i).$$

That is, the continuous function $\sum_{i\in I} b_i(\bar{x}_i^\star, x_i)$ achieves its maximum over the compact $\mathrm{BR}(x)$ at some $\bar{x}^\star \in \mathrm{BR}(x)$. Note that the inequality holds for all ζ of the form considered. Since $x \notin \mathrm{BR}(x)$, $\bar{x}^\star \neq x$, then $\Psi(\bar{x}_i^\star, x_{-i}) < \Psi(x)$ for some $i \in I$, hence we have that $\sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(\bar{x}_i^\star, x_i) < 0$.

Now consider any ζ that is a convex combination of $\zeta_s = \lim_s \nabla\Psi(y^{k_s})$. From the above considerations, we have that

$$\begin{aligned} v^T\zeta &= \alpha_1 v^T\zeta_1 + \cdots + \alpha_l v^T\zeta_l \\ &\le \alpha_1 \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(\bar{x}_i^\star, x_i) + \cdots + \alpha_l \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(\bar{x}_i^\star, x_i) \\ &= \sum_{\substack{i\in I \\ x_i\notin \mathrm{BR}_i(x)}} b_i(\bar{x}_i^\star, x_i) < 0, \end{aligned}$$

for all v and ζ. From here we conclude that $0 \notin \mathcal{L}_F \Psi(x)$, if $x \notin \mathrm{BR}(x)$. Thus, x does not belong to $X \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \mathcal{L}_F \Psi\}}$, which is a contradiction. ■

IV. Self-triggered Communications in Best-Response Dynamics

In this section, we present a sufficient self-triggered communication law as in [13] to lower the frequency at which neighbors' information needs to be updated while still guaranteeing convergence to the set of NE. It is worth highlighting that the amount of neighbors from which information needs to be updated for each player, depends uniquely on the sparsity of the game. That is, our law does not deal with generating a distributed execution of a non-distributed game, but it rather has to do with reducing the time-between-updates. Proofs for all results in this section can be found in Appendix C.

Let $\{t_k^i\}_{k=0}^{\infty} \subseteq \mathbb{R}_{>0}$, such that $t_k^i < t_{k+1}^i$, be the time sequence at which player i updates information about other players, for each $i \in I$. Assume that the i^{th} player has obtained up-to-date information of agent $j \in I \setminus \{i\}$ at some time t_k^i. In what follows, we aim to estimate the largest possible time $t_{k+1}^i > t_k^i$ that an agent i can wait for in order to update information about neighbors while guaranteeing convergence to the set of NE of the game. To do this, we assume that each player has available up-to-date information about its own state at every time $t > t_0^i$. The i^{th} player's action is driven by

$$\dot{x}_i(t) \in \begin{cases} \mathrm{BR}_i(x_{-i}(t_k^i)) - x_i(t), & \text{if } x_i(t_k^i) \notin \mathrm{BR}_i(x_{-i}(t_k^i)), \\ \{0\}, & \text{otherwise}, \end{cases} \tag{4}$$

for time $t \in (t_k^i, t_{k+1}^i]$. See Remark 4.1 about the introduction of zero when $x_i(t_k^i) \in \mathrm{BR}_i(x_{-i}(t_k^i))$.

In the sequel, let us assume that each agent payoff function is Lipschitz continuous with Lipschitz constant $L_i > 0$; that is, $|u_i(x^1) - u_i(x^2)| \leq L_i \|x^1 - x^2\|$, for any $x^1, x^2 \in X$. Let us assume that player i knows L_i. This will help us to compute a self-triggering condition which makes each agent update information whenever its payoff is no longer increasing. First, let us find an upper bound on uncertainty about other player's action with respect to time.

At time t_k^i, player i knows other players' actions, and thus can compute precisely its best-response set, as well as the value of $u_i(x_i^\star(t_k^i), x_{-i}(t_k^i))$, where $x_i^\star(t_k^i) \in \mathrm{BR}_i(x_{-i}(t_k^i))$. Let $j \in I$ be an arbitrary agent $j \neq i$. Let l be such that $t_{l+1}^j > t_k^i \geq t_l^j$ for the given k. Notice that since $\mathrm{BR}_j(x_{-j}(t_l^j))$ is compact, there exists a point $x_j^{\text{fast}} \in \mathrm{BR}_j(x_{-j}(t_l^j))$ such that $x_j^{\text{fast}} \in \operatorname{argmax}_{y \in \mathrm{BR}_j(x_{-j}(t_l^j))} \|y - x_j(t_k^i)\|$. Then, the magnitude of $\dot{x}_j(t)$ defined in (4) is maximized by x_j^{fast}, for all time $t \in (t_k^i, t_{l+1}^j]$ (i.e., $\dot{x}_j(t) = x_j^{\text{fast}} - x_j(t)$ has maximum norm). Assume that $x_j(t_k^i) \notin \mathrm{BR}_j(t_l^j)$. Thus, a fastest solution of (4) for $t \in (t_k^i, t_{l+1}^j]$, is $x_j(t) = x_j^{\text{fast}} - (x_j^{\text{fast}} - x_j(t_k^i))e^{-(t-t_k^i)}$. This implies that the distance $\|x_j(t) - x_j(t_k^i)\|$ is upper bounded by $\|x_j^{\text{fast}} - x_j(t_k^i)\| \left(1 - e^{-(t-t_k^i)}\right)$, for $t \in (t_k^i, t_{l+1}^j]$. However, the i^{th} player does not know the j^{th} player's best-response set, then, the only option is to compute the worst possible case with the available information. Assume that all agents know the action space X. Then, the i^{th} agent can find a point $x_j^{\text{far}}(t_k^i) \in X_j$, which maximizes the distance from the last known position of j. That is, $x_j^{\text{far}}(t_k^i) \in \operatorname{argmax}_{y \in X_j} \|x_j(t_k^i) - y\|$. Then,

$$\|x_j^{\text{fast}} - x_j(t_k^i)\| \left(1 - e^{-(t-t_k^i)}\right) \leq \|x_j^{\text{far}} - x_j(t_k^i)\| \left(1 - e^{-(t-t_k^i)}\right),$$

holds for every $t \in (t_k^i, t_{l+1}^j]$. Since the left-hand side of the inequality above is an upper bound for the movement of

j, for $t \in (t_k^i, t_{l+1}^j]$, we have that

$$\|x_j(t) - x_j(t_k^i)\| \le \|x_j^{\text{far}}(t_k^i) - x_j(t_k^i)\| \left(1 - e^{-(t-t_k^i)}\right). \tag{5}$$

At this point, we can neglect the assumption of $x_j(t_k^i) \notin \text{BR}_j(t_l^j)$, since, if $x_j(t_k^i) \in \text{BR}_j(t_l^j)$, then the solution is trivially $x_j(t) = x_j(t_k^i)$, for $t \in (t_k^i, t_{l+1}^j]$. Therefore, the upper bound in (5) holds.

***Lemma** 4.1:* Inequality (5) holds for all $t_k^i < t$.

Next, we can find an upper bound for $\|x_{-i}(t) - x_{-i}(t_k^i)\|$ as

$$\sum_{j \in I \setminus \{i\}} \|x_j(t_k^i) - x_j^{\text{far}}(t_k^i)\| \left(1 - e^{-(t-t_k^i)}\right) \ge \|x_{-i}(t) - x_{-i}(t_k^i)\|. \tag{6}$$

This upper bound only depends on information available to player i up to time t_k^i. We will use it next to determine the time-update instant t_{k+1}^i.

Next, we use the Lipschitz property of u_i, to obtain

$$u_i(x_i^\star(t_k^i), x_{-i}(t)) \ge u_i(x_i^\star(t_k^i), x_{-i}(t_k^i)) - L_i\|x_{-i}(t) - x_{-i}(t_k^i)\|, \tag{7}$$

and similarly,

$$u_i(x_i(t), x_{-i}(t_k^i)) \le u_i(x_i(t_k^i), x_{-i}(t_k^i)) + L_i\|x_{-i}(t) - x_{-i}(t_k^i)\|. \tag{8}$$

The following lemma states a combination of the bounds in u_i in equations (7), (8), and the bound on $\|x_{-i}(t) - x_{-i}(t_k^i)\|$ from Lemma 4.1, to formulate the self-triggering update condition for each player.

***Lemma** 4.2:* Let $\Gamma = (I, X, u)$ be an ordinal potential game with potential function Φ, fulfilling all properties defined in Theorem 3.1. Assume that u is Lipschitz over X. Let W be a continuous fiber bundle over X such that $W_i(x)$ is compact and convex for all $x \in X$, $i \in I$. Let us consider the self-triggered best-response dynamics as defined in equation (4). Let $\varepsilon > 0$, and suppose $t_k^i > t_0$ is the last time instant when agent i updated information about other agents. Consider any $x_i^\star(t_k^i) \in \text{BR}_i(x_{-i}(t_k^i))$. Let t_{wait}^i be a positive constant. If $t_{k+1}^i > t_k^i$ is such that either

$$u_i(x_i^\star(t_k^i), x_{-i}(t_k^i)) - 2L_i \sum_{j \in I \setminus \{i\}} \|x_j(t_k^i) - x_j^{\text{far}}(t_k^i)\| \left(1 - e^{-(t_{k+1}^i - t_k^i)}\right) = u_i(x_i(t_{k+1}^i), x_{-i}(t_k^i)) + \varepsilon, \tag{9}$$

provided $x_i(t_k^i) \notin \text{BR}_i(x_{-i}(t_k^i))$, or $t_{k+1}^i = t_k^i + t_{\text{wait}}^i$, if $x_i(t_k^i) \in \text{BR}_i(x_{-i}(t_k^i))$, then it holds that $\Phi(x_i(t), x_{-i}(t)) < \Phi(x_i^\star(t_k^i), x_{-i}(t))$ for all $t \in (t_k^i, t_{k+1}^i]$, such that $x_i(t_k^i) \notin \text{BR}_i(x_{-i}(t_k^i))$, and $\Phi(x_i(t), x_{-i}(t)) = \Phi(x_i(t_k^i), x_{-i}(t))$, for all $t \in (t_k^i, t_{k+1}^i]$, if $x_i(t_k^i) \in \text{BR}_i(x_{-i}(t_k^i))$.

Notice that the difference $t_{k+1}^i - t_k^i$ is upper bounded as

$$\begin{aligned} &t_{k+1}^i - t_k^i \\ &\le \max\left\{\log\left(\tfrac{2L_i N \max_{j \in I} \rho(X_j)}{2L_i N \max_{j \in I} \rho(X_j) - L_i \rho(X_i) - \varepsilon}\right), t_{\text{wait}}^i\right\}. \end{aligned}$$

It can be shown that this upper bound follows from (9), but we omit its computation for brevity. This bound will be important to establish precompactness of solutions in the analysis in Section V.

Remark 4.1: Here, we analyze the behavior of player i if its dynamics was given by

$$\dot{x}_i \in \mathrm{BR}_i(x_{-i}(t_k^i)) - x_i(t), \tag{10}$$

for $t \in (t_k^i, t_{k+1}^i]$. If for some time in $(t_k^i, t_{k+1}^i]$, player i is not in its best-response set, the self-triggering time-update policy of Lemma 4.2 guarantees that the payoff at time t is at worst less than the last known best payoff by some $\varepsilon > 0$, provided $x_i(t_k^i) \notin \mathrm{BR}_i(x_{-i}(t_k^i))$. Notice that t_{k+1}^i is the maximum time when this property holds. Then, at t_{k+1}^i information is updated by player i and uncertainty becomes zero again, leading to a new best-response set. This will produce a larger or equal payoff than the current action's payoff. In particular, if the payoff value is the same, then $x_i(t_{k+1}^i) \in \mathrm{BR}_i(x_{-i}(t_{k+1}^i))$.

Suppose a player has reached its best-response set and follows the dynamics (10). Once in $\mathrm{BR}_i(x_{-i}(t_k^i))$, the motion of player i can evolve arbitrarily in the set. In the meantime, the evolution of $\mathrm{BR}_i(x_{-i}(t))$ can lead to a situation where $\mathrm{BR}_i(x_{-i}(t)) \neq \mathrm{BR}_i(x_{-i}(t_k^i))$, while $x_i(t) \in \mathrm{BR}_i(x_{-i}(t))$. In this case, moving toward a point $y \in \mathrm{BR}_i(x_{-i}(t_k^i)) \setminus \mathrm{BR}_i(x_{-i}(t))$ will clearly produce a lower payoff. Thus, the set of velocities that an agent can take needs to be restricted, and it makes necessary to estimate how the best-response set will evolve. Alternatively, one can leverage the fact that $x_i(t_k^i) \in \mathrm{BR}_i(x_{-i}(t_k^i))$, to prescribe the agent velocity to be zero. This motivates the definition of a self-triggered best-response dynamics as in (4), and not as in (10). By means of this, one can guarantee $\Phi(x_i(t_k^i), x_{-i}(t)) = \Phi(x_i(t), x_{-i}(t))$ if and only if $x_i(t) \in \mathrm{BR}_i(x_{-i}(t_k^i))$.

□

Remark 4.2: The self-triggered best-response dynamics in (4) may lead to a zeno-behavior in some examples. That is, as agents approach their best-response sets, they may require information updates more and more often, creating an accumulation point in the time-update sequence. This is a typical trait of general event and self-triggered dynamics. In general, the only way to guarantee a lower bound on the time between updates by this approach is to force it, for example by taking the $(k+1)^{\mathrm{th}}$ update time as $\max\{t_i^k + \Delta t_{\min}, t_i^{k+1}\}$, where $\Delta t_{\min}$ is a small positive number. Introducing this constant is an acceptable trade-off: on the one hand, the nature of the self-triggered approach is still preserved as much as possible, i.e., if possible, communications will be reduced by being triggered at times larger than $\Delta t_{\min}$ until being close to converge, leading to a type of practical convergence. On the other hand, the zeno-behavior is forced to disappear.

□

V. System analysis via invariance theory

In order to formally analyze the self-triggered best-response dynamics, we overapproximate it by means of a larger hybrid system whose solutions include those of interest. To do this, first we associate each agent with a data structure $P^i = (x_i, x_{-i}^i, t_i) \in X \times \mathbb{R}_{\geq 0}$, where $x_{-i}^i = (x_j^i)_{j \in I} \in X_{-i}$ represents the information that agent i maintains on all other agents $j \neq i$, i.e., $x_{-i}^i(t) = x_{-i}(t_k^i)$ for $t \in (t_k^i, t_{k+1}^i]$, and $t_i = t - t_k^i$ for $t \in (t_k^i, t_{k+1}^i]$. Then, $P = (P^1, \dots, P^N) \in (X \times \mathbb{R}_{\geq 0})^N = O$ is an extended state that includes the data structure P^i for each agent. Finally, let us define the projection $\pi : O \to \prod_{i \in I} \mathbb{R}^{n_i}$ as $\pi(P) = (\pi_i(P)) = (x_1, \dots, x_N)$. Using this new notation, we can write the self-triggering condition of Lemma 4.2 as $\Delta_i(P) \leq 0$, where

$$\Delta_i(P) = u_i(x_i^\star, x_{-i}^i) - 2L_i \sum_{j \in I \setminus \{i\}} \|x_j^i - x_j^{\mathrm{far}}\| \left(1 - e^{-t_i}\right) - u_i(x_i, x_{-i}^i) - \varepsilon, \tag{11}$$

with $\|x_j^i - x_j^{\text{far}}\| = \max_{y\in X_j} \|x_j^i - y\|$, and $x_i^\star \in \text{BR}_i(x_{-i}^i)$. We now define a hybrid system on $O = (\mathbb{R}^d \times \mathbb{R})^N$ as follows. First, let $C \subset O$ be the set $C = \cap_{i\in I} C_i = \cap_{i\in I}(\{P \in O \mid \Delta_i(P) \geq 0,\ x_i \in \overline{W_i(x_i, x_{-i}^i) \setminus \text{BR}_i(x_{-i}^i)}\} \cup \{P \in O \mid t_i \leq t_{\text{wait}}^i \text{ and } x_i \in \text{BR}_i(x_{-i}^i)\})$. Secondly, we let $D = \cup_{i\in I}(\{P \in O \mid \Delta_i(P) \leq 0 \text{ and } x_i \in \overline{W_i(x_i, x_{-i}^i) \setminus \text{BR}_i(x_{-i}^i)}\} \cup \{P \in O \mid t_i \geq t_{\text{wait}}^i \text{ and } x_i \in \text{BR}_i(x_{-i}^i)\})$. Define the flow map $F : O \rightrightarrows O$ as $F(P) = \Pi_{i\in I} F_i(P)$, with $F_i(P) = \{(x_i^\star - \pi_i(P), 0, 1) \mid x_i^\star \in \text{BR}_i(x_{-i}^i)\}$, for all $i \in I$. Define the jump map $G : O \rightrightarrows O$ so that $Y \in G(P)$ if and only if $Y^i \in \{P, (x_i, x_{-i}, 0)\}$, for each $i \in I$. Finally, define the hybrid system $\mathcal{H} = (F, G, C, D)$ as

$$\mathcal{H} : \begin{cases} \dot{P} \in F(P), & \text{if } P \in C, \\ P^+ \in G(P), & \text{if } P \in D. \end{cases}$$

Solutions for this system are given by functions $\phi : E \to O$, such that for each $j \in \mathbb{N}$ it holds that $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j = \{t \in \mathbb{R}_{\geq 0} \mid (t, j) \in E\}$, where E is a hybrid domain; see the Appendix E for the definition of this concept. Let $S_{\mathcal{H}}$ be the set of all solutions of $\mathcal{H}$. By definition of $\mathcal{H}$, for each $P \in D$, it holds that $P \in G(P)$. It means that the hybrid system overapproximation generates solutions that remain at the same fixed point P via infinite switching. However, note that these are not solutions of the self-triggered best-response dynamics. Additionally, the set $S_{\mathcal{H}}$ contains trajectories that allow motion inside $\text{BR}_i(x_{-i}^i)$ when x_i has reached $\text{BR}_i(x_{-i}^i)$, see Remark 4.1. Given that Δ_i is a continuous function of P, the sets C and D are closed sets in O. Under the assumption that the u_i are Lipschitz over X, and Φ is component-wise quasiconcave, one can see that F has compact, convex values, it is also locally bounded, and outer semicontinuous in C. The map G is outer semicontinuous by construction.

Let $\Psi = -\Phi$, and consider its extension $\tilde{\Psi} : O \to \mathbb{R}$ defined as $\tilde{\Psi}(P) = \Psi(\pi(P)) = \Psi(x_1, \ldots, x_N)$. In this way, $\tilde{\Psi}$ is a continuous function on O, and a locally Lipschitz function on a neighborhood of C. We now focus on the trajectories of $\mathcal{H}$ whose velocities take values in a subset of the differential inclusion. In other words, we define $\overline{F} : O \rightrightarrows O$ as $\overline{F} = \Pi_{i\in I} \overline{F}_i(P)$, where

$$\overline{F}_i(P) = \begin{cases} (0, 0, 1), & \text{if } x_i \notin \text{int}(\text{BR}_i(x_{-i}^i)), \\ (\text{BR}_i(x_{-i}^i) - x_i, 0, 1), & \text{otherwise.} \end{cases}$$

We have that $\overline{F}(P) \subseteq F(P)$ for all $P \in O$. Note that $\overline{F}$ selects the velocities according to the self-triggered dynamics.

***Lemma** 5.1:* The following holds:

$$\max \mathcal{L}_{\overline{F}} \tilde{\Psi}(P) \leq 0, \quad \forall\, P \in C,$$
$$\max_{P^+ \in G(P)} \tilde{\Psi}(P^+) - \tilde{\Psi}(P) \leq 0, \quad \forall\, P \in D,$$

Moreover, if $P \in C$ is such that for some $i \in I$, $x_i \notin \text{BR}_i(x_{-i}^i)$, then $\max \mathcal{L}_{\overline{F}} \tilde{\Psi}(P) < 0$.

Proof: The condition $\max_{P^+ \in G(P)} \tilde{\Psi}(P^+) - \tilde{\Psi}(P) \leq 0$ holds trivially, as $\tilde{\Psi}(P^+) = \tilde{\Psi}(P)$ for any $P \in O$ and $P^+ \in G(P)$. In order to verify the first condition, we follow along the lines of the proof of Theorem 3.1.

Consider $P \in C$. Any $V \in \overline{F}(P)$ can be written as $V = (V^1, \ldots, V^N)$, where each component V^i has the form $V^i = (x_i^\star - x_i, 0, 1)$, for some $x_i^\star \in \text{BR}_i(x_{-i}^i)$ if $x_i \notin \text{BR}_i(x_{-i}^i)$, or $V^i = (0, 0, 1)$ if $x_i \in \text{BR}_i(x_{-i}^i)$. Let us write $V = P^\star - P$, for an appropriate $P^\star$. Consider any $\zeta \in \partial\tilde{\Psi}(P)$ such that $\zeta = \lim_k \nabla\tilde{\Psi}(Y^k)$, with $Y^k \to P$, and $Y^k \notin \Omega_{\tilde{\Psi}}$. For convenience, let us recall that $\Omega_{\tilde{\Psi}}$ is the set of points at which $\tilde{\Psi}$ is non-differentiable. Since $\tilde{\Psi}$ is independent of the components x_{-i}^i, t_i, note that $\zeta^i = (\pi_i(\zeta), 0, 0)$. If $V^i = (0,0,1)$ for all i, then it holds trivially that $\zeta^T V = 0$. Suppose then that $V^i \neq 0$ for some $i \in I$. Since $Y^k \to P$ we can write $V^T\zeta = \lim_{k\to\infty}(P^\star - Y^k)\nabla\tilde{\Psi}(Y^k)$. Denote and $y^k = \pi(Y^k)$. Using component-wise pseudoconvexity and component-wise quasiconvexity of Ψ, the computations in the proof of Theorem 3.1 can be repeated until we reach that $V^T\zeta \leq \sum_{\substack{i\in I\\ x_i \notin \mathrm{BR}_i(x)}} b_i(x_i^\star, x_i)$, for all $\zeta \in \partial\tilde{\Psi}(P)$.

Now, for $P \in C$, either the condition $\Delta_i(P) \geq 0$ holds for some i or $x_i \in \mathrm{BR}_i(x_{-i}^i)$, for all $i \in I$. Then, by Lemma 4.2 it is true that $\Psi(x_i^\star, x_{-i}) - \Psi(x) \leq 0$, for all $i \in I$. Thus, we have that $V^T\zeta \leq 0$. The result can be extended for any $\zeta \in \partial\tilde{\Psi}(x)$ similarly to Theorem 3.1. Therefore, $\max_{\overline{F}} \mathcal{L}_{\overline{F}}\tilde{\Psi}(P) = \max_{V,\zeta} V^T\zeta \leq 0$.

To prove the second part of this lemma, note that $\max_{\overline{F}} \mathcal{L}_{\overline{F}}\tilde{\Psi}(P) = \max_{V,\zeta} V^T\zeta$, and $\max_{V,\zeta} V^T\zeta \leq \max_{x^\star \in \{y \mid y_i \in \mathrm{BR}_i(x_{-i}^i)\}} \sum_{\substack{i\in I\\ x_i \notin \mathrm{BR}_i(x)}} b_i(x_i^\star, x_i)$. Since $b_i(x_i^\star, x_i)$ is continuous for all $i \in I$, and $\mathrm{BR}_i(x_{-i}^i)$ is a compact set, the right-hand side of the above inequality achieves its maximum at some $\bar{x}_i^\star \in \mathrm{BR}_i(x_{-i}^i)$ for all $i \in I$. Then, if for some $i \in I$, $x_i \notin \mathrm{BR}_i(x_{-i}^i)$, since $P \in C$, it must be that $\Delta_i(P) \geq 0$. By Lemma 4.2, this implies $b_i(x_i^\star, x_i) < 0$ for all $x_i^\star \in \mathrm{BR}_i(x_{-i}^i)$ and, in particular, $b_i(\bar{x}_i^\star, x_i) < 0$. Thus, the strict inequality $\max \mathcal{L}_{\overline{F}}\tilde{\Psi}(P) < 0$ follows. ■

***Theorem** 5.1:* Let $\Gamma = (I, X, u)$ be an ordinal potential game with potential function Φ, fulfilling all properties defined in Theorem 3.1. Assume that u is a Lipschitz continuous function over X. Let W be a continuous fiber bundle over X such that $W_i(x)$ is compact and convex for all $x \in X$, $i \in I$. Let (4) be the self-triggered best-response dynamics for $\mathcal{R}_W(\Gamma)$. Then, all precompact solutions of the self-triggered best-response dynamics converge to the set $X^\star$ of constrained Nash Equilibria.

Proof: Consider a precompact solution ϕ of the self-triggered best response dynamics which, in particular, is a precompact solution of $\mathcal{H}$. Then, the ω-limit set $\Omega(\phi)$ is nonempty, compact, and weakly invariant [21]. Since $\tilde{\Psi}$ satisfies Lemma 5.1, then $\Omega(\phi) \subseteq \tilde{\Psi}^{-1}(r)$ for some r. First, the conditions in Lemma 5.1 imply that $\tilde{\Psi} \circ \phi$ is non-increasing and bounded below. Let r satisfy that $\lim_{t\to\infty, j\to\infty} \tilde{\Psi}(\phi(t,j)) = r$. Take any $P \in \Omega(\phi)$. By definition, $\lim_{l\to\infty} \phi(t_l, j_l) = P$, with $(t_l, j_l) \in E$. Since $\tilde{\Psi}$ is continuous, then it holds that $\lim_{k\to\infty} \tilde{\Psi}(\phi(t_k, j_k)) = \tilde{\Psi}(P) = r$.

Take a $P \in \Omega(\phi)$. Since $\Omega(\phi)$ is weakly forward invariant, there exists a solution to the self-triggered best response dynamics such that $P = \phi(t_0, j_0)$ for some (t_0, j_0). Suppose that there is an i such that for $P_i = (x_i, x_{-i}^i, t_i)$, we have $x_i \notin \mathrm{BR}_i(x_{-i}^i)$. Then, by Lemma 5.1 we have that $\max \mathcal{L}_{\overline{F}}\tilde{\Psi}(P) < 0$. If $P \in C$, and the solution flows, then it must be that $\tilde{\Psi}(\phi(t_0^+, j_0)) < r$, which contradicts $P \in \tilde{\Psi}^{-1}(r)$. Thus, it must be that $P \in \overline{D \setminus C}$. This implies that there exists an i such that either $\Delta_i(P) \leq 0$ or $t_{\mathrm{wait}}^i \leq t_i$. However, after the jump, $\overline{P} = \phi(t_0, j_0 + 1) \in C$. Then, either we have that $\pi_i(\overline{P}) \in \mathrm{BR}_i(\pi(\overline{P}))$, for some $i \in I$, in which case the conclusion follows, or else it will continue flowing afterwards, which leads to a contradiction again. ■

VI. An example: Coverage Control

Throughout this section we describe the problem of optimally deploying a set of N agents in a one-dimensional scenario as a Potential Game. This example will serve to illustrate the proposed coordination strategy using the best-response dynamics and self-triggered control. Proofs for the results in this section can be found in Appendix D.

Let us consider a set of N agents whose movement is constrained to a set $[0, L] \subseteq \mathbb{R}$, where $L > 0$. Thus, all positions of the agents are given by the vector $z = [z_1, \ldots, z_N]^\top \in [0, L]^N$. In order to gather information, each agent i is equipped with a sensor that is able to continuously monitor an interval $[z_i - r_i, z_i + r_i]$, where $r_i > 0$ denotes its effective range. We consider that the sensing action is energy costly, therefore each agent i is able to adjust its own radius r_i limited to a set $[0, \overline{r}_i]$, where $\overline{r}_i$ represents a physical bound on the sensing capacity. We now consider any deployment task as an absolutely continuous map $\varphi : [0, \infty) \to \prod_{i=1}^N ([0, L] \times [0, \overline{r}_i])$, where $\varphi(t) = [z_1(t), r_1(t), \ldots, z_N(t), r_N(t)]^\top$.

In the following, we formulate our deployment problem as a continuous-space game. In order to do this, we let our players be the set of agents to be deployed, the action space $X = \prod_{i=1}^N ([0, L] \times [0, \overline{r}_i])$, their actions are given by the vector $x_i = [z_i, r_i]^\top \in X_i \subset \mathbb{R}^2$. Let us also define the set $Q_i(x) = \{j \in I \setminus \{i\} \mid |z_i - z_j| < r_i + r_j\}$. Notice that $Q_i(x)$ is the set of agents that are sensing areas also monitored by i, and that $i \in Q_j(x)$ if and only if $j \in Q_i(x)$. Let us split the set $Q_i(x)$ in the sets $Q_i^l(x) = \{j \in Q_i(x) \mid j < i\}$, and $Q_i^r(x) = \{j \in Q_i(x) \mid j > i\}$. Define the set $W = \{x \in X \mid [z_i - r_i, z_i + r_i] \not\subset (z_j - r_j, z_j + r_j)$ for any $i, j \in I\} \cap \{x \in X \mid z_j \leq z_i,$ if $j \in Q_i^l(x)\} \cap \{x \in X \mid z_i \leq z_j,$ if $j \in Q_i^r(x)\}$. We let the payoff functions be given by

$$u_i(x) = 2r_i - \alpha \sum_{j \in Q_i(z,r)} v(z_i, z_j, r_i, r_j) - \alpha \, \mathrm{loss}_i(x_i)$$

for $i \in I$, where

$$\mathrm{loss}_i(x_i) = \max\{z_i + r_i - L, 0\} + \max\{z_i - r_i, 0\},$$

and $v(z_i, z_j, r_i, r_j)$ is defined as:

$$v(z_i, z_j, r_i, r_j) = \max\{\min\{z_i + r_i, z_j + r_j\} - \max\{z_i - r_i, z_j - r_j\}, 0\},$$

for $i \in I$, $j \in Q_i(x)$. The parameter α is given to weight the sensor overlapping in the payoff function. On the other hand, the value of $v(z_i, z_j, r_i, r_j)$ is equal to the length of the interval sensed simultaneously by i and j. In the following, we show that this is an exact potential game.

***Lemma** 6.1:* The deployment game stated above is an exact potential game with potential function:

$$\Phi(x) = \frac{1}{2} \sum_{i=1}^N (u_i(x) + 2r_i - \alpha \, \mathrm{loss}_i(x_i)). \tag{12}$$

***Lemma** 6.2:* The set W is compact and convex.

Next, we analyze the concavity properties of Φ over W.

***Lemma** 6.3:* The potential function Φ of the agent deployment game is component-wise concave in W.

Given that concavity implies pseudoconcavity and also quasiconcavity, this game holds the properties on Φ that are required on Theorem 5.1. From (25) we can also see that Φ is a linear combination of pointwise maximum of linear functions; hence a locally Lipschitz function of its argument. Furthermore, since W is a compact set, then Φ is globally Lipschitz on W.

***Lemma** 6.4:* Define $\mathrm{ub}_i = z_{i+1} - r_{i+1}$, and $\mathrm{lb}_i = z_{i-1} + r_{i-1}$, for $i = 2, \ldots, N-1$, $\mathrm{ub}_1 = z_2 - r_2$ and $\mathrm{lb}_1 = 0$, $\mathrm{ub}_N = L$ and $\mathrm{lb}_N = z_{N-1} + r_{N-1}$. If $\alpha > 1$, the best-response set of the i^{th} player is given by:

- If $2\overline{r}_i > \mathrm{ub}_i - \mathrm{lb}_i$ and $0 \leq \frac{\mathrm{lb}_i + \mathrm{ub}_i}{2} \leq L$.

$$\mathrm{BR}_i(x_{-i}) = \{x_i\} = \left\{ \left(\frac{\mathrm{lb}_i + \mathrm{ub}_i}{2}, \frac{|\,\mathrm{ub}_i - \mathrm{lb}_i\,|}{2} \right) \right\}.$$

- If $2\overline{r}_i > \mathrm{ub}_i - \mathrm{lb}_i$ and $0 > \frac{\mathrm{lb}_i + \mathrm{ub}_i}{2}$

$$\mathrm{BR}_i(x_{-i}) = \{x_i\} = \{(0, -\mathrm{ub}_i)\}\,.$$

- If $2\overline{r}_i > \mathrm{ub}_i - \mathrm{lb}_i$ and $L < \frac{\mathrm{lb}_i + \mathrm{ub}_i}{2}$

$$\mathrm{BR}_i(x_{-i}) = \{x_i\} = \{(0, \mathrm{lb}_i - L)\}\,.$$

- If $2\overline{r}_i \leq \mathrm{ub}_i - \mathrm{lb}_i$

$$\{x_i \in W_i(x_i, x_{-i}) \mid \mathrm{lb}_i + \overline{r}_i \leq z_i \leq \mathrm{ub}_i - \overline{r}_i, r_i = \overline{r}_i\}.$$

Next, we present some simulations to gain a better understanding of the solution's performance.

The simulation scenario consists of a set of 3 agents, which are covering a 1-dimensional set $[0, 10]$. Each agent i has a maximum action radius $\overline{r}_i = 2$, and follows the self-triggered communication law as established in Theorem 5.1. The Lipschitz constant for the self-triggered updates is $L_i = 1$, and we take $t^i_{\text{wait}} = 1$ second. We use a first order Euler method to integrate the equation. Our simulation horizon is 10 seconds, the step time chosen for integration is 2.5×10^{-4} seconds. We use a value $\Delta t_{\min} = 0.1$. Further, in order to simulate the differential inclusion, at each time we pick one element of the best response set with uniform probability distribution. Figure 1 shows the actual

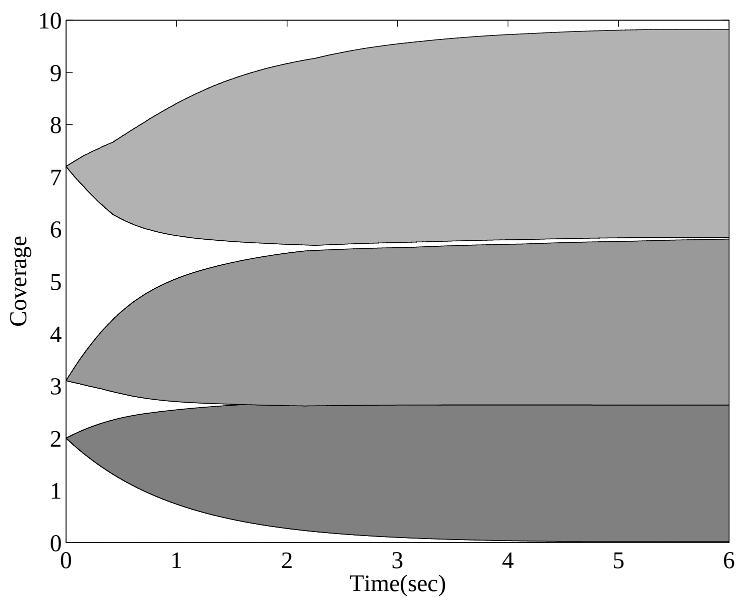

Fig. 1. Coverage vs. time. It is possible to tell which area is covered by each agent. They are organized from bottom to top from 1 through 3.

coverage provided by each agent. Notice that since player 3 stops as soon as it gets into its best-response set, there is a segment near the upper end of the set that is uncovered. This does not happen in the lower end, because agent 1 never reaches its maximum radius, then, whenever the upper bound of this player's sensing region touches the lower bound of agent 2 sensing region, its position is going to move down to the middle value of the interval $[0, z_2 - r_2]$, while the radius is increasing towards $\frac{z_2 - r_2}{2}$. It is worth to point out that the achieved Nash Equilibrium is not optimal, because it does not guarantee full coverage. Clearly, an action profile such as $\sum_{i=1}^{3} r_i = 10$, $z_1 = r_1$, $z_2 = 2r_1 + r_2$ and $z_3 = 10 - r_3$ is an optimal Nash Equilibrium for this game, because it guarantees full coverage, showing zero overlapping. Figure 2 shows the update times used by each agent to get information about the current state of the

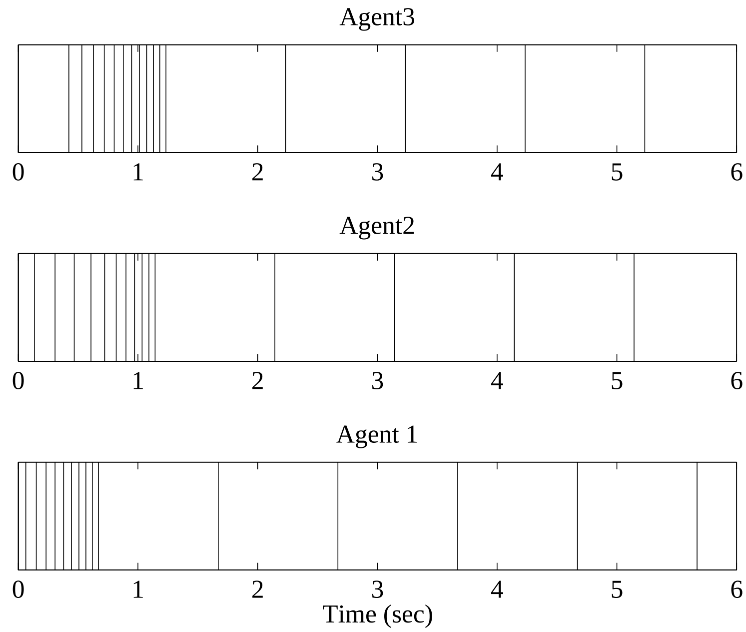

Fig. 2. Information update times for each agent.

game. We can see that when players approach their best-response sets, update times tend to accumulate, however, due to the parameter $\Delta t_{\min}$, this accumulation does not happen, and in practice, the agent does not move away from the best-response set. Thus, the parameter $\Delta t_{\min}$ is an effective countermeasure against Zeno-behavior, as it has been discussed in Remark 4.2. Finally, Figure 3 demonstrates the theoretical result on the monotonic increase

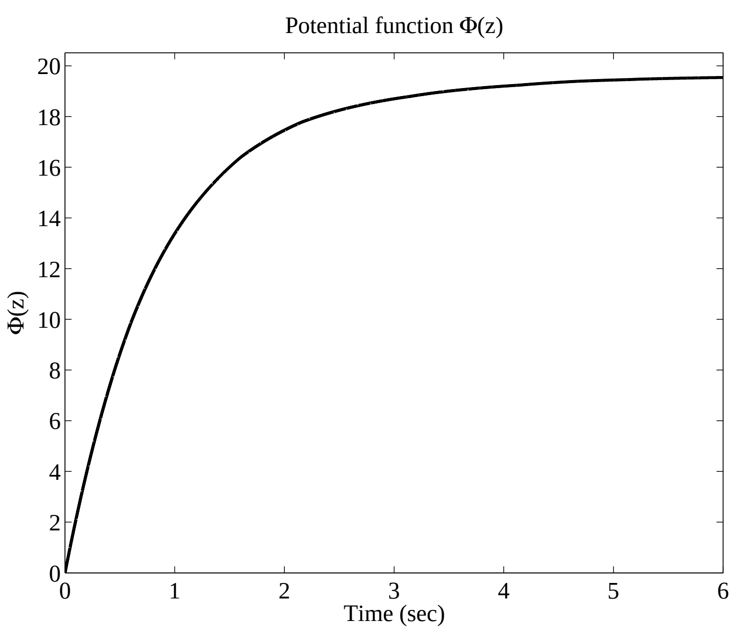

Fig. 3. Potential function Φ vs. time.

of Φ along the solutions of the self-triggered best-response dynamics.

VII. Conclusions

In this work, we characterize the convergence properties of the continuous time best-response dynamics for a continuous-action-space potential game, with N players and n_i-dimensional action space for each player i. We show under general conditions that all solutions of the best-response dynamics of a potential game will converge to the set of Nash equilibria set of the game. With the aim of making the best-response dynamics more practical, a self-triggered communication strategy is proposed to reduce communications among agents while still guaranteeing convergence to the desired configurations. The self-triggered best-response dynamics is modeled as a hybrid system, and convergence analysis is made using analysis tools in [21].

As an example, we characterize the 1D deployment of a mobile sensor network as a potential game, where we implement the self-triggered best-response dynamics. The objective of the game, is to maximize the sensor overall coverage, avoiding footprint overlapping. Simulations show in a more intuitive way all theoretical results. Future work will be devoted to study the effects of delays in the self-triggered best response dynamics.

Appendix

A. Some definitions on real analysis

Definition *1.1:* [22] A function $V : \mathbb{R}^n \to \mathbb{R}$ is called regular at x if:

- for all $v \in \mathbb{R}^n$ the right directional derivative $V'(x, v)$ exists:
- for all $v \in \mathbb{R}^n$, $V'(x, v) = V_\circ(x, v) = \liminf_{y \to x, h \to 0^+} \frac{V(y+hv)-V(y)}{h}$.

Definition *1.2:* [23] The set-valued map $F : X \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is upper semicontinuous at a point $x_0 \in X$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, $F(x) \subseteq F(x_0) + B_\delta(0)$.

In order to analyze the convergence of the best-response dynamics to the set of equilibria, we must introduce some additional definitions related to stability analysis [22], [24].

Definition *1.3:* Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. Let Ω_f be the set of points at which f is not differentiable. By Rademacher's theorem, Ω_f is a zero-measure set with respect to the Lebesgue measure. The generalized gradient $\partial f : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ of f is given by

$$\partial f(x) \triangleq \mathrm{co}(\{\lim_{i\to\infty} \nabla f(x_i) \mid x_i \to x, x_i \notin \Omega_f\}). \tag{13}$$

As seen in [20], [22], [21], the stability of a system defined by a differential inclusion can be analyzed by means of non-continuously differentiable Lyapunov functions and generalized invariance principles based on set-valued Lie derivatives. We recall these tools next.

Definition *1.4:* Given a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and a set-valued map $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, the set-valued Lie derivative $\mathcal{L}_G f : \mathbb{R}^d \rightrightarrows \mathbb{R}$ of f with respect to G is given by

$$\mathcal{L}_G f(x) = \{a \in \mathbb{R} \mid \exists v \in G(x) \text{ such that } \zeta^\top v = a, \text{ for all } \zeta \in \partial f(x)\}. \tag{14}$$

B. LaSalle's invariance principle for differential inclusions and nonsmooth Lyapunov functions

The following theorem is an extension of the classical LaSalle's invariance principle.

Theorem *1.1 ([20], [22]):* Let $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a locally bounded, upper semicontinuous set-valued map which takes nonempty, compact and convex values, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function, according to the definition of regularity in [20] (see also the Appendix). Let S be a compact and strongly invariant set for $\dot{x} \in G(x)$, and assume that $\max \mathcal{L}_G f(y) \leq 0$ for each $y \in S$. Then, all solutions $x : [0, \infty) \to \mathbb{R}^d$ of $\dot{x} \in G(x)$ converge to the largest weakly invariant set M contained in

$$S \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \mathcal{L}_G f\}}.$$

□

C. *Proofs of results on Section IV*

1) Proof of lemma 4.1: We have already showed that the upper bound holds for $t \in (t_k^i, t_{l+1}^j]$. Next we show that the inequality also holds for $t > t_{l+1}^j$. With a slight abuse of notation, let $x_j^{\text{fast}} \in \text{BR}_j(x_j(t_{l+1}^j))$ denote a point such that $x_j^{\text{fast}} \in \text{argmax}_{y \in \text{BR}_j(x_{-j}(t_{l+1}^j))} \|y - x_j(t_k^i)\|$. Assume that $x_j(t_{l+1}^i) \notin \text{BR}_j(t_{l+1}^j)$. In this way, the solution of (4) which maximizes $\|x_j(t) - x_j(t_k^i)\|$, for all $t \in (t_{l+1}^j, t_{l+2}^j]$ is given by $x_j^{\text{fast}} - \left(x_j^{\text{fast}} - x_j(t_{l+1}^j)\right) e^{-(t - t_{l+1}^j)}$. Therefore, we have

$$\|x_j(t) - x_j(t_k^i)\| \leq \|x_j^{\text{fast}} - \left(x_j^{\text{fast}} - x_j(t_{l+1}^j)\right) e^{-(t - t_{l+1}^j)} - x_j(t_k^i)\|, \tag{15}$$

for $t > t_{l+1}^j$. By adding and subtracting $x_j(t_k^i) e^{-(t - t_{l+1}^j)}$ inside the norm of the right-hand side of (15), and then using the triangular inequality, we obtain

$$\begin{aligned} \|x_j^{\text{fast}} - \left(x_j^{\text{fast}} - x_j(t_{l+1}^j)\right) e^{-(t - t_{l+1}^j)} - x_j(t_k^i)\| \leq & \|x_j^{\text{fast}} - x_j(t_k^i)\| \left(1 - e^{-(t - t_{l+1}^j)}\right) \\ & + \|x_j(t_{l+1}^j) - x_j(t_k^i)\| e^{-(t - t_{l+1}^j)}. \end{aligned} \tag{16}$$

Now, from (15) and (16) we derive

$$\|x_j(t) - x_j(t_k^i)\| \leq \|x_j^{\text{fast}} - x_j(t_k^i)\| \left(1 - e^{-(t - t_{l+1}^j)}\right) + \|x_j(t_{l+1}^j) - x_j(t_k^i)\| e^{-(t - t_{l+1}^j)}.$$

Applying (5) with $t = t_{l+1}^j$, and then replacing $\|x_j(t_{l+1}^j) - x_j(t_k^i)\|$ in the inequality above, we have

$$\begin{aligned} \|x_j(t) - x_j(t_k^i)\| \leq & \|x_j^{\text{fast}} - x_j(t_k^i)\| \left(1 - e^{-(t - t_{l+1}^j)}\right) \\ & + \|x_j^{\text{far}}(t_k^i) - x_j(t_k^i)\| \left(1 - e^{-(t_{l+1}^j - t_k^i)}\right) e^{-(t - t_{l+1}^j)}. \end{aligned} \tag{17}$$

Notice that the following equality holds for all t

$$\begin{aligned} \|x_j^{\text{far}}(t_k^i) - x_j(t_k^i)\| \left(1 - e^{-(t - t_k^i)}\right) = & \|x_j^{\text{far}}(t_k^i) - x_j(t_k^i)\| \left(1 - e^{-(t - t_{l+1}^j)}\right) \\ & + \|x_j^{\text{far}}(t_k^i) - x_j(t_k^i)\| \left(1 - e^{-(t_{l+1}^j - t_k^i)}\right) e^{-(t - t_{l+1}^j)}. \end{aligned} \tag{18}$$

From the definition of $x_j^{\text{far}}(t_k^i)$, we have that $\|x_j^{\text{far}}(t_k^i) - x_j(t_k^i)\| \geq \|x_j^{\text{fast}} - x_j(t_k^i)\|$. Therefore, the first term of the right-hand side of (18) is an upper bound for the first term of the right-hand side of (17), while both expressions' second terms of the right-hand side are equal. This implies that the left-hand side of (18) is an upper bound of $\|x_j(t) - x_j(t_k^i)\|$ for $t_{l+1}^j < t < t_{l+2}^j$.

Now, we can see that if $x_j(t_{l+1}^j) \in \text{BR}_j(t_{l+1}^j)$, the unique solution is $x_j(t) = x_j(t_{l+1}^j)$, and the computed upper bound holds trivially.

The same reasoning holds for $t > t_{l+m}^j$, for $m \in \mathbb{N}$, and the result follows. ■

2) Proof of lemma 4.2: At time t_k^i, player i has exact information about other players' actions. Assume that $x_i(t_k^i) \notin \mathrm{BR}_i(x_{-i}(t_k^i))$, then $u_i(x_i^\star(t_k^i), x_{-i}(t_k^i)) > u_i(x_i(t_k^i), x_{-i}(t_k^i))$. Using equations (7), (6), we can obtain a lower bound for $u_i(x_i^\star(t_k^i), x_{-i}(t))$ as follows

$$u_i(x_i^\star(t_k^i), x_{-i}(t)) \geq u_i(x_i^\star(t_k^i), x_{-i}(t_k^i)) - L_i \sum_{j \in I \setminus \{i\}} \|x_j(t_k^i) - x_j^{\mathrm{far}}(t_k^i)\| \left(1 - e^{-(t - t_k^i)}\right). \tag{19}$$

Similarly, using equations (8) and (6), an upper bound for $u_i(x_i(t), x_{-i}(t))$ is found as follows

$$u_i(x_i(t), x_{-i}(t)) \leq u_i(x_i(t), x_{-i}(t_k^i)) + L_i \sum_{j \in I \setminus \{i\}} \|x_j(t_k^i) - x_j^{\mathrm{far}}(t_k^i)\| \left(1 - e^{-(t - t_k^i)}\right). \tag{20}$$

In order to guarantee that $u_i(x_i^\star(t_k^i), x_{-i}(t)) \geq u_i(x_i(t), x_{-i}(t))$ holds, it is sufficient to ensure that the right-hand side of (20) is less or equal than the right-hand side of (19). That is, if

$$u_i(x_i^\star(t_k^i), x_{-i}(t)) - 2L_i \sum_{j \in I \setminus \{i\}} \|x_j(t_k^i) - x_j^{\mathrm{far}}(t_k^i)\| \left(1 - e^{-(t - t_k^i)}\right) \geq u_i(x_i(t), x_{-i}(t_k^i)) + \varepsilon. \tag{21}$$

for some positive and sufficiently small ε, which in particular, must satisfy $\varepsilon < u_i(x_i^\star(t_k^i), x_{-i}(t_k^i)) - u_i(x_i(t_k^i), x_{-i}(t_k^i))$, then $u_i(x_i^\star(t_k^i), x_{-i}(t_k^i)) > u_i(x_i(t_k^i), x_{-i}(t_k^i))$. Next, notice that if i did not update its information about other players, then $x_i(t)$ would approach $\mathrm{BR}_i(x_{-i}(t_k^i))$ as $t \to \infty$, according to the differential inclusion in (4). Therefore, the value $u_i(x_i(t), x_{-i}(t_k^i))$ would converge to $u_i(x_i^\star(t_k^i), x_{-i}(t_k^i))$. However, the second term at the left-hand side of (21) is negative and decreasing on time, while the whole expression is continuous on time. Therefore, for each $\varepsilon > 0$ there exists a finite time for which (21) does not hold anymore. The first time for which the equality happens is given by t_{k+1}^i in (9). Then, for $t \in [t_k^i, t_{k+1}^i]$, it holds that $u_i(x_i^\star(t_k^i), x_{-i}(t)) > u_i(x_i(t), x_{-i}(t))$. Since Γ is a potential game, this inequality holds if and only if

$$\Phi(x_i^\star(t_k^i), x_{-i}(t)) > \Phi(x_i(t), x_{-i}(t)). \tag{22}$$

The second part of the theorem, assuming that $x_i(t_k^i) \in \mathrm{BR}_i(x_{-i}(t_k^i))$ holds trivially, given that by i^{th} player dynamics, $x_i(t) = x_i(t_k^i)$ for $t \in (t_k^i, t_{k+1}^i]$. ■

D. Results on the agent deployment example

1) Proof of lemma 6.1: First, note that we can rewrite $\Phi(x)$, for $x = (x_i, x_{-i}) \in X$, as follows:

$$\begin{aligned} \Phi(x) = \sum_{k=1}^{N} 2r_k - \frac{1}{2}\alpha \sum_{\substack{k=1 \\ k \neq i}}^{N} \sum_{\substack{j \in Q_k(x) \\ j \neq i}} v(z_k, z_j, r_k, r_j) \\ - \alpha \sum_{j \in Q_i(x)} v(z_i, z_j, r_i, r_j) - \alpha \sum_{k=1}^{N} \mathrm{loss}_k(x_k). \end{aligned} \tag{23}$$

Now, let us take $x' = (x'_i, x_{-i})$, with $x'_i = [z'_i, r'_i]$ and compute $\Phi(x) - \Phi(x')$ for some $i \in I$. Using the expression (23), the difference simplifies to:

$$\begin{aligned}\Phi(x) - \Phi(x') =& 2r_i - \alpha \sum_{j \in Q_i(x)} v(z_i, z_j, r_i, r_j) - 2r'_i + \alpha \sum_{j \in Q_i(x')} v(z'_i, z_j, r'_i, r_j) \\ & - \alpha \, \mathrm{loss}_i(x_i) + \alpha \, \mathrm{loss}_i(x'_i) \\ =& u_i(x) - u_i(x').\end{aligned} \tag{24}$$

This shows that Φ is a potential function for the deployment game we have defined above, and thus the latter becomes an exact potential game. ■

2) Proof of lemma 6.2: In order to prove convexity, we recast the conditions for x to be in W as follows:

- For every $i \in I$, and for every $j \in Q_i^l(x)$, $z_i - r_i \geq z_j - r_j$, $z_i + r_i \geq z_j + r_j$, and $z_j \leq z_i$.
- For every $i \in I$, and for every $j \in Q_i^r(x)$, $z_i - r_i \leq z_j - r_j$, $z_i + r_i \leq z_j + r_j$, and $z_i \leq z_j$.

Define $x, x' \in W$, as: $x = [z_1, r_1, \dots, z_N, r_N]^\top$, and $x' = [z'_1, r'_1 \dots, z'_N, r'_N]^\top$. From the inequalities above we have that if $j \in Q_i^l(x)$, $sz_i - sr_i \geq sz_j - sr_j$, and $(1-s)(z'_i - r_i) \geq (1-s)(z'_j - r'_j)$, for any $s \in [0,1]$. Summing up these inequalities, we obtain $sz_i + (1-s)z'_i - (sr_i + (1-s)r'_i) \geq sz_j + (1-s)z'_j - (sr_j + (1-s)r'_j)$. Following the same procedure for all inequalities, (involving also those for $j \in Q_i^r(x)$), we have that $sx + (1-s)x' \in W$. Thus, W is convex. Finally, compactness follows directly from the definition of W as a closed subset of X. ■

3) Proof of lemma 6.3: If $x \in W$, the maps $v(z_i, z_j, r_i, r_j)$ can be expressed in a more straightforward form, and the potential function Φ rewritten as follows:

$$\begin{aligned}\Phi(x) =& \sum_{k=1}^{N} 2r_k + 2r_i - \frac{1}{2}\alpha \sum_{\substack{k=1 \\ k \neq i}}^{N} \sum_{\substack{j \in Q_k(x) \\ j \neq i}} v(z_k, z_j, r_k, r_j) - \alpha \sum_{j=1}^{i} \max\{z_j + r_j - (z_i - r_i), 0\} \\ & - \alpha \sum_{j=i}^{N} \max\{z_i + r_i - (z_j - r_j), 0\} - \alpha \sum_{k=1}^{N} \mathrm{loss}_k(x_k),\end{aligned} \tag{25}$$

for any $i \in I$. The expressions $\max\{z_j + r_j - (z_i - r_i), 0\}$, and $\max\{z_i + r_i - (z_j - r_j), 0\}$ come from simplifying $v(z_i, z_j, r_i, r_j)$ on W. First, note that the map v becomes $z_j + r_j - (z_i - r_i)$ if $j \in Q_i^l(x)$, or $z_i + r_i - (z_j - r_j)$ if $j \in Q_i^r(x)$, if $x \in W$. Since for each $j \notin Q_i(x)$, $v(z_k, z_j, r_k, r_j) \leq 0$, we use the maximum between v and 0, in order to have summation indices that are independent from x. The expressions $\max\{z_j + r_j - (z_i - r_i), 0\}$ and $\max\{z_i + r_i - (z_j - r_j), 0\}$ are the point-wise maxima of linear maps of x_i, thus their sum is a convex function of x_i, for every $i \in I$. The same occurs with the function $\mathrm{loss}_i(x_i)$. Due to the negative sign affecting these sums, Φ becomes a concave function of x_i for all $i \in I$; thus it is component-wise concave on W. ■

4) Proof of lemma 6.4: First, notice that we can find an upper bound for the payoff function $u_i(x)$ as

$$u_i(x) \leq 2r_i - \alpha \max\{0, z_i + r_i - \mathrm{ub}_i\} - \alpha \max\{0, \mathrm{lb}_i - (z_i - r_i)\}.$$

Let $x_i^\dagger$ be an element of the set we want to prove is the best-response set. Let us analyze the first case. Assume that $2\overline{r}_i > \mathrm{ub}_i - \mathrm{lb}_i$ and $0 \leq \frac{\mathrm{lb}_i + \mathrm{ub}_i}{2} \leq L$. In addition, assume that $\mathrm{ub}_i \geq \mathrm{lb}_i$. We obtain a payoff $u_i(x_i^\dagger, x_{-i}) = \mathrm{ub}_i - \mathrm{lb}_i$. Then, consider a deviation off the defined best-response set, this is, a vector $\gamma = [\gamma_z, \gamma_r]^\top \in \mathbb{R}^2$, where $\|\gamma\|$ is arbitrarily small, and $x_i^\dagger + \gamma \in W_i(x_i^\dagger + \gamma, x_{-i})$. Then, we obtain

$$u_i(x_i^\dagger + \gamma, x_{-i}) \leq \mathrm{ub}_i - \mathrm{lb}_i + 2\gamma_r - \alpha \max\{0, \gamma_z + \gamma_r\} - \alpha \max\{0, -\gamma_z + \gamma_r\}.$$

If $\gamma_r < 0$, clearly $u_i(x_i^\dagger + \gamma, x_{-i}) < u_i(x_i^\dagger, x_{-i})$. If $\gamma_r = 0$, then $u_i(x_i^\dagger + \gamma, x_{-i}) \leq \mathrm{ub}_i - \mathrm{lb}_i - \alpha|\gamma_z|$. If $\gamma_r > 0$, and $\gamma_r \geq |\gamma_z|$, then, $\gamma_r + \gamma_z \geq 0$, and also $\gamma_r - \gamma_z \geq 0$, therefore $u_i(x_i^\dagger + \gamma, x_{-i}) \leq \mathrm{ub}_i - \mathrm{lb}_i - 2(\alpha - 1)\gamma_r$. If $\gamma_r > 0$, and $\gamma_r < |\gamma_z|$, then either $\gamma_r + \gamma_z$, or $\gamma_r - \gamma_z$ is less than zero, but the other is greater than zero, then $u_i(x_i^\dagger + \gamma, x_{-i}) \leq \mathrm{ub}_i - \mathrm{lb}_i - 2\alpha(\gamma_r + |\gamma_z|)$, leading to $u_i(x_i^\dagger + \gamma, x_{-i}) \leq u_i(x_i^\dagger, x_{-i})$.

Now, let us assume that $\mathrm{lb}_i > \mathrm{ub}_i$. In this case, $u_i(x_i^\dagger + \gamma, x_{-i}) \leq (1 - \alpha)(\mathrm{lb}_i - \mathrm{ub}_i)$. Notice that the only choices of γ such that $x_i^\dagger + \gamma \in W_i(x_i^\dagger + \gamma, x_{-i})$ are those that hold $\gamma_z \leq \gamma_r$, $\gamma_z + \gamma_r \geq 0$ which imply that $\gamma_r \geq 0$. The upper bound for the payoff is given by

$$\begin{aligned} u_i(x_i^\dagger + \gamma, x_{-i}) \leq & \mathrm{ub}_i - \mathrm{lb}_i + 2\gamma_r - \alpha \max\{0, \mathrm{lb}_i - \mathrm{ub}_i + \gamma_z + \gamma_r\} \\ & - \alpha \max\{0, \mathrm{lb}_i - \mathrm{ub}_i - \gamma_z + \gamma_r\}. \end{aligned}$$

Since we choose γ arbitrarily small, and $\mathrm{lb}_i - \mathrm{ub}_i > 0$, it holds that $u_i(x_i^\dagger + \gamma, x_{-i}) \leq (1 - \alpha)(\mathrm{lb}_i - \mathrm{ub}_i + 2\gamma_r)$. We have seen that for each particular case, regardless of the choice of γ the payoff for the best-response singleton greater than any of its neighborhood, then by concavity of u_i, the maximum is global, and the result follows.

The proof for second and third cases, is analogous to that of the first case, then we omit it.

For the last case, the proof is rather simple. just consider that by structure of the payoff functions, $u_i(x_i^\dagger, x_{-i}) = 2\bar{r}_i \geq u_i(x_i, x_{-i})$ for all $x_i \in W_i(x_i, x_{-i})$, $i \in I$. Then, any choice of r_i different from $\bar{r}_i$ leads to a lower payoff. Finally, consider any point x_i such that $r_i = \bar{r}_i$, but either $\mathrm{lb}_i + \bar{r}_i < z_i$, or $z_i > \mathrm{ub}_i - \bar{r}_i$. It leads the maximum functions that determine the upper bound for the payoff, to be positive, and then the payoff is smaller. ∎

E. Notions of set-valued analysis and hybrid systems analysis

The following two definitions are taken from [21].

***Definition** 1.5:* The set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous at a point $x_0 \in X$ if for every sequence of points x_k convergent to x and any convergent sequence of points $y_k \in F(x_k)$, it holds that $y \in F(x)$, where $y = \lim_{k\to\infty} y_k$. The map is outer semicontinuous if it is outersemicontinuous at each $x \in \mathbb{R}^n$. Given a set $X \subset \mathbb{R}^n$, F is outer semicontinuous relative to X if the set-valued map defined by $F(x)$ for $x \in X$ and $\emptyset$ for $x \notin X$ is outer semicontinuous at each $x \in X$.

***Definition** 1.6:* A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}, j]), \tag{26}$$

for some finite sequences of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$. Moreover, E is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain.

The hybrid system $\mathcal{H}$ is well-posed if it satisfies the following basic conditions:

- C and D are closed subsets of O;
- F is outer semicontinuous (see definition of outer semicontinuity in Appendix) and locally bounded relative to $C \subseteq O$, and $F(P)$ is convex for every $P \in C$;

- G is outer semicontinuous and locally bounded relative to $D \subseteq O$.

References

[1] C. Harris, "On the rate of convergence of continuous-time fictitious play," *Games and Economic Behavior*, vol. 22, no. 2, pp. 238–259, 1998.

[2] M. Benaïm, J. Hofbauer, and S. Sorin, "Stochastic approximations and differential inclusions," *SIAM Journal on Control and Optimization*, vol. 44, no. 1, pp. 328–348, 2005.

[3] J. Hofbauer and S. Sorin, "Best response dynamics for continuous zero-sum games," *Discrete and Continuous Dynamical Systems Series B*, vol. 6, no. 1, p. 215, 2006.

[4] E. Barron, R. Goebel, and R. Jensen, "Best response dynamics for continuous games," *Proceedings of the American Mathematical Society*, vol. 138, no. 3, pp. 1069–1083, 2010.

[5] A. MacKenzie and L. DaSilva, "Game theory for wireless engineers," *Synthesis Lectures on Communications*, vol. 1, no. 1, pp. 1–86, 2006.

[6] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks.* American Mathematical Society, Princeton University Press, 2009. Available at http://www.coordinationbook.info.

[7] J. Marden, G. Arslan, and J. Shamma, "Cooperative control and potential games," *IEEE Transactions on Systems, Man, & Cybernetics. Part B: Cybernetics*, vol. 39, no. 6, pp. 1393–1407, 2009.

[8] M. Zhu and S. Martínez, "Distributed coverage games for energy-aware mobile sensor networks," *SIAM Journal on Control and Optimization*, vol. 51, no. 1, pp. 1–27, 2013.

[9] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, 2010.

[10] E. Garcia and P. Antsaklis, "Model-based event-triggered control for systems with quantization and time-varying network delays," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 422–434, 2013.

[11] X. Wang and M. Lemmon, "Event-triggering in distributed networked systems with data dropouts and delays," *Hybrid systems: Computation and control*, pp. 366–380, 2009.

[12] P. Wan and M. Lemmon, "Event-triggered distributed optimization in sensor networks," in *Symposium on Information Processing of Sensor Networks*, (San Francisco, CA), pp. 49–60, 2009.

[13] C. Nowzari and J. Cortés, "Self-triggered coordination of robotic networks for optimal deployment," *Automatica*, vol. 48, no. 6, pp. 1077–1087, 2012.

[14] M. Zhong and C. Cassandras, "Asynchronous distributed optimization with event-driven communication," *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2735–2750, 2010.

[15] X. Wang, Y. Sun, and N. Hovakimyan, "Asynchronous task execution in networked control systems using decentralized event-triggering," *Systems & Control Letters*, vol. 61, no. 9, pp. 936–944, 2012.

[16] A. Cortés and S. Martínez, "Self-triggered, best-response dynamics for sensor coverage," in *American Control Conference*, (Washington, DC), pp. 6385–6390, June 2013.

[17] D. Monderer and L. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, pp. 124–143, 1996.

[18] J. Hofbauer and K. Sigmund, "Evolutionary game dynamics," *Bulletin of the American Mathematical Society*, vol. 40, no. 4, p. 479, 2003.

[19] E. Ok, *Real analysis with economic applications*, vol. 10. Princeton University Press, 2011.

[20] A. Bacciotti and F. Ceragioli, "Stability and stabilization of discontinuous systems and nonsmooth lyapunov functions.," *ESAIM: Control, Optimisation & Calculus of Variations*, vol. 4, pp. 361–376, 1999.

[21] R. Goebel, R. Sanfelice, and A. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, 2009.

[22] J. Cortés, "Discontinuous dynamical systems - a tutorial on solutions, nonsmooth analysis, and stability," *IEEE Control Systems Magazine*, vol. 28, no. 3, pp. 36–73, 2008.

[23] J. Aubin and H. Frankowska, *Set-valued analysis.* Birkhäuser, 1990.

[24] A. Filippov, *Differential Equations with Discontinuous Right-hand side.* Kluwer Academic Publishers, 1988.