Singularly perturbed filters for dynamic average consensus

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***Abstract*— This paper proposes two continuous-time dynamic average consensus filters for networks described by balanced and weakly-connected directed topologies. Our distributed filters, termed 1st-Order-Input (*FOI-DCF*) and 2nd-Order-Input Dynamic (*SOI-DCF*) Consensus Filters, respectively, allow agents to track the average of their dynamic inputs within an $O(\epsilon)$-neighborhood. The convergence results and stability analysis rely on singular perturbation theory for non-autonomous systems. The only requirement on the set of reference inputs involves continuous bounded derivatives, up to the second derivative for *FOI-DCF* and up to the third derivative for *SOI-DCF*. For the special case of dynamic inputs offset by a static value, we show that *SOI-DCF* converges to the exact dynamic average with no steady-state error. Numerical examples show how the proposed algorithms closely track the average of dynamic inputs.**

I. Introduction

This paper deals with the dynamic average consensus problem for a network of agents. This problem involves designing an algorithm for each agent which tracks the average of the network agents' time-varying inputs using only local and neighboring agents' information. In recent years, the dynamic average consensus problem of multi-agent systems has attracted increasing attention due to its broad application in areas such as multi-robot coordination [1], distributed estimation [2], sensor fusion [3], [4], and distributed tracking [5].

The work [6] proposes a dynamic average consensus algorithm which is able to track the average of ramp reference inputs with zero steady-state error. In addition to the limited inputs that it can track, the filter is not robust to estimator initialization errors. Instead, [3] proposes a low-pass consensus filter which tracks, with a steady state error, the average of identical inputs with a uniformly bounded rate. The work [7] also proposes a consensus filter which achieves dynamic average consensus over a common time-varying reference signals. However, the algorithm assumes that agents know the nonlinear model which generates the time-varying reference signals. Using input-to-state stability analysis, [8] proposes a proportional dynamic average consensus algorithm that can track the average of bounded reference inputs with bounded derivatives with bounded steady-state error. This approach is generalized in [9] to achieve robust dynamic average consensus of a broad class of time-varying inputs, assuming model information about them is available when designing the filter.

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The stability and performance of the aforementioned algorithms are studied in a continuous-time setting. The work [10] develops instead a discrete-time dynamic average consensus estimator. The convergence analysis for the proposed dynamic average consensus algorithms relies upon the input-to-output stability property of discrete-time static average consensus algorithms in the presence of external disturbances. With a proper initialization of the internal states, the proposed estimator can track, with bounded steady state error, the average of the time-varying inputs whose nth-order difference is bounded. In the special case where the nth-order difference is asymptotically zero, the estimates of the average converge to the true average asymptotically with one time step delay. The conditions on the initialization, similarly to [6], makes the results in [10] not robust to initialization errors.

In this paper, we propose two novel continuous-time dynamic average consensus algorithms that allow a group of agents to track the average of their reference inputs within a $O(\epsilon)$-neighborhood. We term these algorithms 1*st-Order-Input Dynamic Consensus Filter* (*FOI-DCF*) and 2*nd-Order-Input Dynamic Consensus Filter* (*SOI-DCF*), respectively. These filters perform the task starting from any finite initial condition when the interaction network is described by a balanced and weakly-connected directed graph. The convergence and stability analysis relies on an original application to dynamic consensus problems of singular perturbation theory of non-autonomous systems. The only requirement on the set of reference inputs involves continuous and bounded derivatives of the inputs (up to second-order derivatives for *FOI-DCF* and up to third-order derivatives for *SOI-DCF*). For the special case of dynamic inputs offset by a static value, we show that *SOI-DCF* converges to the exact dynamic average with no steady-state error. For static inputs, both filters converge to the exact input average. Numerical examples show good tracking performance of the algorithms. Simulations also show that the proposed filters are robust to sporadic switching network topologies and tracking inputs that are not necessarily differentiable. This is a reasonable expected behavior if dwelling times are sufficiently large. Additionally, the proposed filters are able to handle, with no extra adjustments, permanent changes in the network topology due to agents joining or leaving the network. One can attribute this robustness properties to the global stability and convergence properties of the proposed filters.

The paper organization is as follows. Section II introduces the main notation used throughout the paper and reviews the static consensus filter proposed in [8]. Section III motivates the use of singular perturbation theory to generate dynamic consensus algorithms. Section IV contains the main results on the proposed dynamic average consensus algorithms. To demonstrate the effectiveness of the proposed filters, Section V presents various numerical examples. For the convenience of the reader, the Appendix gives a brief review of the Singular Perturbation problem definition and the main result we use to characterize the properties of our dynamic consensus filters.

II. Notation and Preliminaries

The vector $\mathbf{1}_n$ represents an n-dimensional vector with all elements equal to one, and $\mathbf{I}_n$ represents the identity matrix with dimension n. We denote by $\mathbf{A}^T$ the transpose of matrix $\mathbf{A}$. For matrices $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$, we let $\mathbf{A} \otimes \mathbf{B}$ denote their Kronecker product. We denote $\delta_1(\epsilon) \in O(\delta_2(\epsilon))$ if there exist positive constants c and k such that

$$|\delta_1(\epsilon)| \leq k|\delta_2(\epsilon)|, \quad \forall\, |\epsilon| < c.$$

For time-varying variables, e.g. $\mathbf{v}(t)$, most of the time, we drop the time dependency; only when the emphasis on the time dependency is necessary we will use the complete notation.

In the following, we present some basic notions from algebraic graph theory (for more details see [11] or [12]). A directed graph, or simply digraph, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \cdots, N\}$ is a finite set called the vertex set and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the edge set. A graph is undirected if $(u, v) \in \mathcal{E}$ anytime $(v, u) \in \mathcal{E}$. We denote by $\mathcal{A} \in \mathbb{R}^{N\times N}$ the *adjacency* matrix of $\mathcal{G}$, with the property that $a_{ij} > 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The *out-degree* and *in-degree* of a vertex $v_i,\ i \in \{1, \cdots, N\}$, are respectively, $d_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$ and $d_{in}(v_i) = \sum_{j=1}^{N} a_{ji}$. The out-degree matrix D_{out} is the diagonal matrix defined by $(\mathbf{D}_{out})_{ii} = d_{out}(v_i)$, for all $i \in \{1, \cdots, N\}$. The *Laplacian* matrix is $\mathbf{L} = \mathbf{D}_{out} - \mathcal{A}$. Note that $\mathbf{L}\mathbf{1}_N = 0$. An undirected graph is called *connected* if, for every pair of vertices in $\mathcal{V}$, there is a path that has them as its end vertices. The digraph is called *weakly connected* if it is connected when viewed as an undirected graph, that is, a disoriented digraph. Here, a digraph is called *balanced* if, for every vertex, the in-degree and out-degree are equal. The network considered here is composed of $N > 1$ agents whose communication topology is described by a weakly-connected and balanced digraph. We use $\mathcal{D}$ to refer to such a graph. For this type of networks, we have $\mathrm{rank}(\mathbf{L}) = N-1$, $\mathbf{1}_N^T\mathbf{L} = 0$, and $\mathbf{L} + \mathbf{L}^T$ is a positive semi-definite matrix.

For network problems involving internal states of dimension $n > 1$, we define $\boldsymbol{\mathcal{L}}_n = \mathbf{L} \otimes \mathbf{I}_n$. The local variables at each agent are distinguished by a superscript i, e.g., $\mathbf{u}^i(t)$ is the local dynamic input of agent i. If $\mathbf{p}^i \in \mathbb{R}^n$ is a local vector at agent i, the aggregated $\mathbf{p}^i$'s of the network is represented by $\mathbf{p}_T = \begin{bmatrix}\mathbf{p}^{1\,T} & \cdots & \mathbf{p}^{N\,T}\end{bmatrix}^T \in \mathbb{R}^{nN}$.

We use the following lemma from the literature to develop our main results:

Lemma 2.1 (Static average consensus filter [8]): Given a networked system with topology $\mathcal{D}$, and constant inputs $\mathbf{u}^i \in \mathbb{R}^n$, for $i = 1, \cdots, N$, consider the following solver at each agent i:

$$\dot{\mathbf{z}}^i = -(\mathbf{z}^i - \mathbf{u}^i) - \sum_{j=1}^{N} \mathbf{L}_{ij}(\mathbf{z}^j + \boldsymbol{\nu}^j),$$
$$\dot{\boldsymbol{\nu}}^i = \sum_{j=1}^{N} \mathbf{L}_{ij}\mathbf{z}^j.$$

From any initial condition $\mathbf{z}(0) \in \mathbb{R}^n$ and $\boldsymbol{\nu}(0) \in \mathbb{R}^n$, $\mathbf{z}^i$ converges to $\frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i$ exponentially.

III. Motivation to Use Singular Perturbation Theory to Generate Consensus Algorithms

In this section, we motivate the use of singular perturbation theory in devising distributed dynamic average consensus algorithms. The simplest dynamics that generates $\mathbf{x}^i(t) \to \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)$, asymptotically, in each agent is the following:

$$\dot{\mathbf{x}}^i = -(\mathbf{x}^i - \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)) + \frac{1}{N}\sum_{i=1}^{N}\dot{\mathbf{u}}^i(t).$$

To decentralize this dynamics, we can make use of a mechanism that generates the average of the inputs and also the average of the derivative of inputs, *rapidly*, in each agent in a distributed fashion. Then, the distributed dynamic consensus algorithm becomes a two-time scale operation, a fast dynamics to generate each average and a slow dynamics to track the input average. A singularly perturbed dynamical system is an appropriate platform to construct such an algorithm. The aforementioned fast and slow dynamics could be realized by means of the following—seemingly more straightforward and systematic—mixed discrete/continuous-time algorithm running synchronously at each node $i \in \{1, \cdots, N\}$:

(Initialization) at $k = 0$ initialize $\mathbf{x}^i(0) \in \mathbb{R}^n$
while data exists **do**
  Obtain inputs $\mathbf{u}^i(k)$ and $\dot{\mathbf{u}}^i(k)$
  Initialize $\mathbf{z}^i(0), \boldsymbol{\nu}^i(0) \in \mathbb{R}^n$
  Solve the following dynamical equation, until it reaches its equilibrium point

$$\begin{cases} \epsilon\,\dot{\mathbf{z}}^i(t) = -(\mathbf{z}^i(t) + \mathbf{u}^i(k) + \dot{\mathbf{u}}^i(k)) \\ \qquad\qquad - \sum_{i=j}^{N}\mathbf{L}_{ij}(\mathbf{z}^j(t) + \boldsymbol{\nu}^j(t)), \\ \epsilon\,\dot{\boldsymbol{\nu}}^i(t) = \sum_{j=1}^{N}\mathbf{L}_{ij}\mathbf{z}^j(t), \end{cases} \quad (1)$$

  Let $\mathbf{z}^i$ converge to equilibrium $\bar{\mathbf{z}}^i$
  Define:

$$\mathbf{x}^i(k+1) = \mathbf{x}^i(k) - \Delta t\left(\mathbf{x}^i(k) + \bar{\mathbf{z}}^i(k)\right) \quad (2)$$

  $k \leftarrow k + 1$
end while

In the above algorithm Δt is the time-step and $0 < \epsilon \ll 1$ is a scalar value. Note that using the result of Lemma 2.1, at each time step k, the dynamical system (1) acts as a static consensus filter with static input $\mathbf{u}^i(k)+\dot{\mathbf{u}}^i(k)$. This filter converges exponentially to $\bar{\mathbf{z}}^i(k)=-\frac{1}{N}\sum_{i=1}^N(\mathbf{u}^i(k)+\dot{\mathbf{u}}^i(k))$. For very small ϵ, the convergence rate of (1) is high. Therefore, at any time-step k, (2) becomes:

$$\mathbf{x}^i(k+1)=\mathbf{x}^i(k)-\Delta t\left(\mathbf{x}^i(k)-\frac{1}{N}\sum_{i=1}^N(\mathbf{u}^i(k)+\dot{\mathbf{u}}^i(k))\right),$$

for $i=1,\cdots,N$. For small Δt, the stability and convergence of the above difference equation can be studied using the following continuous-time model (represented in compact form):

$$\dot{\mathbf{y}}_T=-\mathbf{y}_T, \tag{3}$$

where $\mathbf{y}_T=\mathbf{x}_T-\mathbf{1}_N\otimes(\frac{1}{N}\sum_{i=1}^N\mathbf{u}^i)$. The dynamical system (3) is a stable linear system with all eigenvalues equal to -1. Therefore, (3) converges to zero exponentially. As a result, $\mathbf{x}^i$ in (2) converges to $\frac{1}{N}\sum_{i=1}^N\mathbf{u}^i(t)$ exponentially, for all $i=1,\cdots,N$.

The aforementioned algorithm performs dynamic average consensus with an exponential convergence rate. However, this algorithm is a conceptual algorithm; the cost of solving (1) at each time-step is the main drawback of this algorithm which virtually makes it un-implementable. Inspired by the multi time-scale structure observed above, we next make use of singular perturbation theory to weave together steps 5–7 and devise a continuous-time dynamic average consensus filter. By doing so, we avoid solving the fast dynamical system at each iteration of the algorithm, i.e., the slow dynamics does not need to wait for the fast dynamics to converge.

IV. Dynamic Continuous-time Consensus Filters via Singularly Perturbed Dynamics

Here, we employ singular perturbation theory to construct new continuous-time dynamic average consensus algorithms. The main advantage of these filters is their convergence irrespectively of the initial condition and, therefore, their robustness against initialization errors across the network. Furthermore, the convergence of the filters does not require any knowledge of the dynamics generating the inputs.

Consider the following distributed filters, listed based on the complexity of the structure:

- 1*st-Order-Input Dynamic Consensus Filter* (*FOI-DCF*):

for $i=1,\cdots,N$

$$\begin{cases}\epsilon\,\dot{\mathbf{z}}^i=-(z^i+\beta\,\mathbf{u}^i+\dot{\mathbf{u}}^i)-\sum_{i=j}^N\mathbf{L}_{ij}(\mathbf{z}^j+\boldsymbol{\nu}^j),\\ \epsilon\,\dot{\boldsymbol{\nu}}^i=\sum_{j=1}^N\mathbf{L}_{ij}\mathbf{z}^j,\end{cases} \tag{4a}$$

$$\dot{\mathbf{x}}^i=-\beta\,\mathbf{x}^i-\mathbf{z}^i, \tag{4b}$$

where β is a positive scalar.

- 2*nd-Order-Input Dynamic Consensus Filter* (*SOI-DCF*):

for $i=1,\cdots,N$

$$\begin{cases}\epsilon\,\dot{\mathbf{z}}^i=-(\mathbf{z}^i+\beta\,\mathbf{u}^i+\dot{\mathbf{u}}^i)-\sum_{j=1}^N\mathbf{L}_{ij}(\mathbf{z}^j+\boldsymbol{\nu}^j)\\ \qquad -\epsilon(\beta\,\dot{\mathbf{u}}^i+\ddot{\mathbf{u}}^i),\\ \epsilon\,\dot{\boldsymbol{\nu}}^i=\sum_{j=1}^N\mathbf{L}_{ij}\mathbf{z}^j,\end{cases} \tag{5a}$$

$$\dot{\mathbf{x}}^i=-\beta\,\mathbf{x}^i-\mathbf{z}^i, \tag{5b}$$

where β is a positive scalar.

In the following, we show that these filters generate an $O(\epsilon)$-approximation of the average of the dynamic inputs $\frac{1}{N}\sum_{i=1}^N\mathbf{u}^i(t)$, in a distributed fashion, for networks with graph topology $\mathcal{D}$.

Theorem 4.1 (Convergence of FOI-DCF): Consider a networked system with topology $\mathcal{D}$. If the first and the second derivatives of the input signal u^i at each agent are continuous and bounded for $t\geq 0$, then there is a small enough $\epsilon^\star>0$ such that, for all $\epsilon\in(0,\epsilon^\star]$, the trajectories of the filter *FOI-DCF*, starting from any finite initial conditions, satisfy $\|\mathbf{x}^i(t)-\frac{1}{N}\sum_{i=1}^N\mathbf{u}^i(t)\|<O(\epsilon)$ in finite time.

Theorem 4.2 (Convergence of SOI-DCF): Consider a networked system with topology $\mathcal{D}$. If the first, the second, and the third derivatives of the input signal u^i at each agent are continuous and bounded for $t\geq 0$, then there is a small enough $\epsilon^\star>0$ such that, for all $\epsilon\in(0,\epsilon^\star]$, the trajectories of the filter *SOI-DCF*, starting from any finite initial conditions, satisfy $\|\mathbf{x}^i(t)-\frac{1}{N}\sum_{i=1}^N\mathbf{u}^i(t)\|<O(\epsilon)$ in finite time.

The proof of Theorems 4.1 and 4.2 are very similar, therefore, we present them together.

Proof of Theorems 4.1 and 4.2: The proof is based on showing the filters satisfy the conditions of Theorem A.1 globally. The boundary layer dynamics (fast dynamics) for both *FOI-DCF* and *SOI-DCF* is (for $i=1,\cdots,N$)

$$\begin{cases}\frac{d\,\mathbf{z}^i}{d\tau}=-(\mathbf{z}^i+\beta\,\mathbf{u}^i(t)+\dot{\mathbf{u}}^i(t))-\sum_{i=1}^N\mathbf{L}_{ij}(\mathbf{z}^j+\boldsymbol{\nu}^j),\\ \frac{d\,\boldsymbol{\nu}^i}{d\tau}=\sum_{i=1}^N\mathbf{L}_{ij}\mathbf{z}^j.\end{cases} \tag{6}$$

Invoking Lemma 2.1, this fast dynamics converges to the following values for each filter exponentially and globally:

$$\mathbf{z}^i=-\frac{1}{N}\sum_{i=1}^N(\beta\,\mathbf{u}^i+\dot{\mathbf{u}}^i),\quad i=1,\cdots,N. \tag{7}$$

Substituting (7) into (4b) (similarly, in (5b)), we obtain the following reduced system (slow dynamics), for the filters *FOI-DCF* and *SOI-DCF*:

$$\dot{\mathbf{x}}^i=-\beta\,\mathbf{x}^i+\frac{1}{N}\sum_{i=1}^N(\beta\,\mathbf{u}^i+\dot{\mathbf{u}}^i),\quad i=1,\cdots,N. \tag{8}$$

Consider the following change of variables:

$$\mathbf{y}^i=\mathbf{x}^i-\frac{1}{N}\sum_{i=1}^N\mathbf{u}^i,\quad i=1,\cdots,N.$$

In the new coordinates, (8) is represented as

$$\dot{\mathbf{y}}^i = -\beta\,\mathbf{y}^i, \quad i = 1, \cdots, N. \tag{9}$$

For $\beta > 0$, the system (9) is a stable linear system with system matrix eigenvalues equal to $-\beta$. Thus, (9) converges globally exponentially to zero, which is equivalent to $\mathbf{x}^i$ exponentially converging to $\frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)$, $i = 1, \cdots, N$.

Both filters, based on the corresponding required conditions for input signals, satisfy the differentiability and Lipschitz conditions of Theorem A.1 on any compact set of $(\mathbf{x}_T, \mathbf{z}_T, \boldsymbol{\nu}_T)$. Thus, all the conditions of Theorem A.1 are satisfied globally, and the estimates of (17), (18) and (19) hold, for all $t \geq 0$ and for any bounded initial states. Note that the slow dynamics at each agent is globally exponentially approaching the average $\frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)$, therefore, $\|\mathbf{x}_T(t) - \mathbf{1}_N \otimes \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)\| < O(\epsilon)$ in a finite time, and as a result $\|\mathbf{x}^i(t) - \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)\| < O(\epsilon)$ in finite time. ■

The slow and fast dynamical analysis of the filters *FOI-DCF* and *SOI-DCF* is exactly the same. The guaranteed convergence bound is also of the same order. In the following, we show that the filter *SOI-DCF* has advantages over *FOI-DCF*, at the price of a slight increase in the complexity of the filter, and the extra condition on the input signals. For example, one can show that for *SOI-DCF*, we have $\sum_{i=1}^{N}\mathbf{x}^i(t) \to \sum_{i=1}^{N}\mathbf{u}^i(t)$ as $t \to \infty$, for any $\epsilon > 0$. The main advantage of the filter *SOI-DCF* over *FOI-DCF* is stated in the following result.

Corollary 4.1 (SOI-DCF for inputs offset by a static value): Consider a networked system with topology $\mathcal{D}$ as in Theorem 4.1, subject to similar assumptions on the input signals of agents. When the difference in the input signals is a static offset, i.e., $\mathbf{u}^i(t) = \mathbf{u}(t) + \bar{\mathbf{u}}^i$ where $\bar{\mathbf{u}}^i$ is a constant vector, the filter *SOI-DCF* converges to the exact input average with no steady-state error. That is, $\mathbf{x}^i(t) \to \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)$, for any $\epsilon > 0$.

Proof: Consider the following change of the variables:

$$\begin{cases}\mathbf{p}^i = \mathbf{z}^i + \beta\,\mathbf{u} + \dot{\mathbf{u}}, \\ \mathbf{q}^i = \mathbf{x}^i - \mathbf{u},\end{cases}, \quad i = 1, \cdots, N.$$

Then, we can re-write (5), as follows (compact form)

$$\begin{cases}\epsilon\,\dot{\mathbf{p}}_T = -(\mathbf{p}_T + \beta\,\bar{\mathbf{u}}_T) - \boldsymbol{\mathcal{L}}_n(\mathbf{p}_T + \boldsymbol{\nu}_T), \\ \epsilon\,\dot{\boldsymbol{\nu}}^i(t) = \boldsymbol{\mathcal{L}}_n\mathbf{p}_T,\end{cases} \tag{10a}$$

$$\dot{\mathbf{q}}_T = -\beta\,\mathbf{q}_T - \mathbf{p}_T. \tag{10b}$$

We can show that the equilibrium points of this system are at ($\bar{\mathbf{p}}^i = -\frac{\beta}{N}\sum_{i=1}^{N}\bar{\mathbf{u}}^i$, $\bar{\mathbf{q}}^i = \frac{1}{N}\sum_{i=1}^{N}\bar{\mathbf{u}}^i$, $\bar{\boldsymbol{\nu}}^i$) for $i = 1, \cdots, N$, where

$$-\boldsymbol{\mathcal{L}}_{\mathbf{n}}\bar{\boldsymbol{\nu}}_T = (\bar{\mathbf{p}}_T + \beta\,\bar{\mathbf{u}}_T).$$

Consider the following Lyapunov function, where $\tilde{\mathbf{q}}_T = \mathbf{q}_T - \bar{\mathbf{q}}_T$, $\tilde{\mathbf{p}}_T = \mathbf{p}_T - \bar{\mathbf{p}}_T$, and $\tilde{\boldsymbol{\nu}}_T = \boldsymbol{\nu}_T - \bar{\boldsymbol{\nu}}_T$:

$$V = \frac{\beta}{2}\tilde{\mathbf{q}}_T^2 + \frac{\epsilon}{8}\tilde{\mathbf{p}}_T^2 + \frac{\epsilon}{8}\tilde{\boldsymbol{\nu}}_T^2.$$

The derivative of this Lyapunov function along the trajectories of (10) is

$$\dot{V} = -\frac{1}{4}\tilde{\mathbf{p}}_T^T(\boldsymbol{\mathcal{L}}_n + \boldsymbol{\mathcal{L}}_n^T)\tilde{\mathbf{p}}_T - (\frac{1}{2}\tilde{\mathbf{p}}_T - \beta\,\tilde{\mathbf{q}}_T)^2,$$

which is negative semi-definite. It is zero in the set $\mathcal{S} = \{\tilde{\mathbf{p}}_T, \tilde{\mathbf{q}}_T, \tilde{\boldsymbol{\nu}}_T \in \mathbb{R}^{nN} \,|\, \tilde{\mathbf{p}}_T = \mathbf{1}_N \otimes \boldsymbol{\alpha},\ \tilde{\mathbf{p}}_T = 2\beta\,\tilde{\mathbf{q}}_T\}$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$. We can show that $\{\tilde{\mathbf{q}}_T = \mathbf{0},\ \tilde{\mathbf{q}}_T = \mathbf{0},\ \tilde{\boldsymbol{\nu}}_T = \mathbf{1}_N \otimes \boldsymbol{\gamma}\}$, where $\boldsymbol{\gamma} \in \mathbb{R}^n$, is the smallest invariant set contained in $\mathcal{S}$. From the LaSalle Invariance Principle it now follows that $\tilde{\mathbf{q}}_T \to \mathbf{0}$ as $t \to \infty$. This results in $\mathbf{x}^i(t) \to \mathbf{u}(t) + \frac{1}{N}\sum_{i=1}^{N}\bar{\mathbf{u}}^i = \frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)$ as $t \to \infty$, for $i = 1, \cdots, N$, in the filter *SOI-DCF*. ■

Remark 4.1 (Convergence for static inputs): Following an argument similar to the proof of Corollary 4.1, we can show that, for static inputs, for any $\epsilon > 0$, both proposed filters converge to the exact input average with no steady-state error. □

Remark 4.2 (Role of β): As it is shown in the proof of Theorems 4.1, and 4.2, β is the rate of convergence of the slow dynamics. By choosing a large β, we can increase the rate of convergence. However, to keep the two time-scale structure of the filters, β has to be chosen smaller relative to ϵ^{-1}. Quantifying the convergence neighborhood and the effect of β on the size of this neighborhood is left as a future work. Furthermore, one should notice that β is a global variable known to all agents in the network. To guarantee convergence to the right dynamic input average of $\frac{1}{N}\sum_{i=1}^{N}\mathbf{u}^i(t)$, in its $O(\epsilon)$ neighborhood, every agent should agree upon a common value for β before running the consensus filters. The following filter allows the agents to use different values for β and still converge to the right dynamic average. However, note that to provide this robustness with respect to β, we are requiring extra communication channels.

$$\text{for } i = 1, \cdots, N, \tag{11a}$$

$$\begin{cases}\epsilon\,\dot{\mathbf{z}}^i = -(\mathbf{z}^i + \mathbf{u}^i) - \sum_{i=j}^{N}\mathbf{L}_{ij}(\mathbf{z}^j + \boldsymbol{\nu}^j), \\ \epsilon\,\dot{\boldsymbol{\nu}}^i = \sum_{j=1}^{N}\mathbf{L}_{ij}\mathbf{z}^j,\end{cases} \tag{11b}$$

$$\begin{cases}\epsilon\,\dot{\mathbf{y}}^i = -(\mathbf{y}^i + \dot{\mathbf{u}}^i) - \sum_{i=j}^{N}\mathbf{L}_{ij}(\mathbf{y}^j + \boldsymbol{\mu}^j), \\ \epsilon\,\dot{\boldsymbol{\mu}}^i = \sum_{j=1}^{N}\mathbf{L}_{ij}\mathbf{y}^j,\end{cases} \tag{11c}$$

$$\dot{\mathbf{x}}^i = -\beta^i\,\mathbf{x}^i - \beta^i\,\mathbf{z}^i - \mathbf{y}^i, \tag{11d}$$

where β^i's are all positive scalars. To guarantee convergence to the $O(\epsilon)$ neighborhood of the dynamic input average, the requirement on the input signals is the boundedness and continuity of their first and second derivatives. Notice that using different β^i results in different convergence rates at each agent. Then one can expect that the tracking is not coherent across the network agents. The stability and convergence analysis of this filter is along the same lines of the proof of Theorem 4.1 and omitted for brevity. □

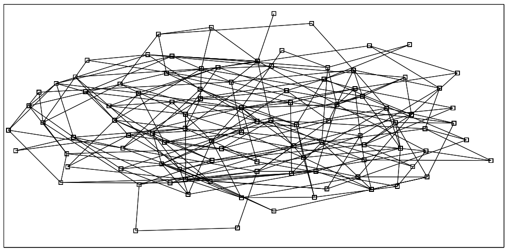


Fig. 1. Network

V. Numerical Examples

In order to give the reviewers a demonstration of the different aspects of the proposed filters, in the following we provide an extensive set of numerical examples. However, in the final version of the paper, due to the space limitation, we need to reduce the number of examples. Here, we mainly perform simulations using the filter *FOI-DCF*. A more comprehensive set of examples including simulations conducted using the filter *SOI-DCF* can be found in [13].

In the following simulations, the thick dashed line is the dynamic input average and the thin colored lines are the time histories of the local x^i states of the filters. In all the simulations below, all the initial conditions for the proposed filters here are selected uniformly randomly in $\mathcal{U}[-2, 2]$.

A. Example 1 (Performance evaluation of the filter FOI-DCF over a large network)

Consider the randomly generated undirected network (using Matlab BGL package [14]) shown in Fig. 1 which consists of $N = 100$ agents. The local input signals are

$$u^i(t) = a_i \sin(b_i\, t + c_i), \quad i = 1, \cdots, N,$$

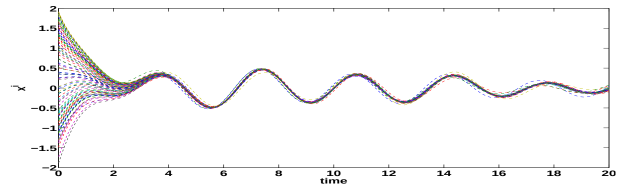
where the input coefficients are generated randomly uniformly in the following ranges: $a_i \sim \mathcal{U}[-5, 5]$, $b_i \sim \mathcal{U}[1, 2]$, $c_i \sim \mathcal{U}[0, \pi/2]$.

Figures 2 shows the time histories of the local internal states x^i generated by the filter *FOI-DCF* and the dynamic input average for the three cases of ($\epsilon = 0.01$, $\beta = 1$), ($\epsilon = 0.001$, $\beta = 1$) and ($\epsilon = 0.001$, $\beta = 3$). As expected, the smaller the ϵ is, the better the tracking, and the larger the β, the faster the convergence to the $O(\epsilon)$ neighborhood is.

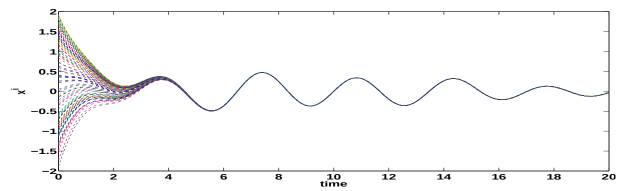
B. Example 2 (Comparison between the filter FOI-DCF and the FODAC algorithm of [10])

Consider the weakly-connected and balanced directed graph in Fig. 3 . The input signals are

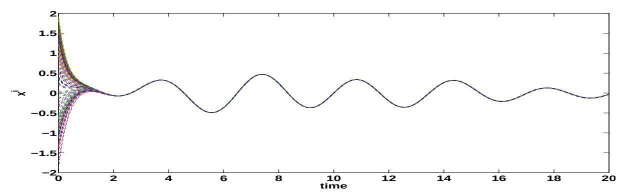
$$\begin{cases} u^1(t) = 5\,\sin(t) + \frac{10}{t+2} + 1,\; u^2(t) = 5\,\sin(t) + \frac{10}{(t+2)^2} + 1, \\ u^3(t) = 5\,\sin(t) + \frac{10}{(t+2)^3} + 1, \\ u^4(t) = 5\,\sin(t) + 10\,\mathrm{e}^{-t} + 4,\; u^5(t) = 5\,\sin(t). \end{cases}$$



(a) $\epsilon = 0.01$ and $\beta = 1$



(b) $\epsilon = 0.001$ and $\beta = 1$



(c) $\epsilon = 0.001$ and $\beta = 3$

Fig. 2. Simulation results for example 1

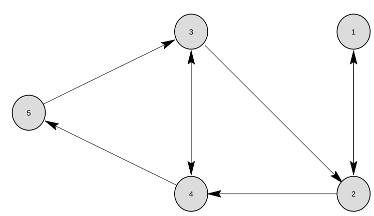
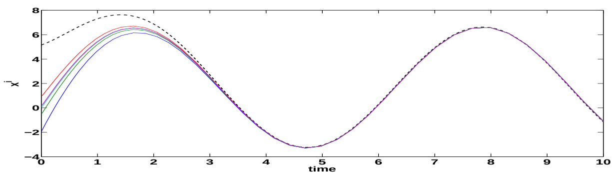
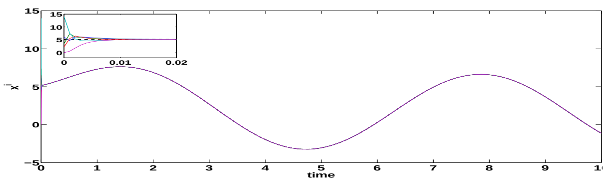


Fig. 3. Network of example 2

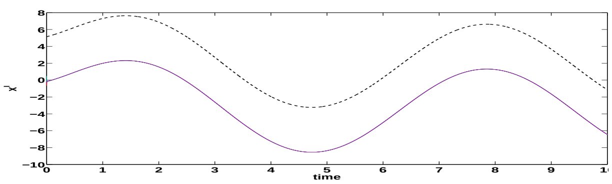
In this example, we compare the performance of the proposed filter *FOI-DCF* with the First-Order Dynamic Average Consensus (FODAC) algorithm of [10]. Figures 4 shows the result of the simulations. To generate the simulations for FODAC algorithm of [10], we used the step size $h = 0.001$ which is higher than the bound for guaranteed convergence (to accelerate the simulations). As Fig. 4 shows, the FODAC algorithm of [10] starting from the required initial conditions of $x^i(0) = u^i(-h)$, $i = 1, \cdots, N$, has a higher rate of convergence than the filter *FOI-DCF*. However, as demonstrated in Fig. 3(c), for initial conditions other than $x^i(0) = u^i(-h)$ this filter produces a large steady sate error. In this simulation, for the filter *FOI-DCF*, we used $\epsilon = 0.01$ and $\beta = 1$. In case of agent failure, the FODAC algorithm requires an adjustment of the initial condition in order to guarantee tracking. As we see in the following this adjustment is necessary in our simulated algorithms.



(a) Filter *FOI-DCF* with $\epsilon = 0.01$ and $\beta = 1$



(b) The FODAC algorithm of [10] with $x^i(0) = u^i(-h)$



(c) The FODAC algorithm of [10] with $x^i(0) \in \mathcal{U}[-2, 2]$

Fig. 4. Simulation results for example 2.

C. Example 3 (Performance of the filter FOI-DCF over a network with time-varying topology)

Consider the time-varying weakly-connected and balanced directed graph in Fig. 5. The input signals are

$$\begin{cases} u^1(t) = 6\,\cos(0.2\,t),\; u^2(t) = \mathrm{atan}(0.2\,t), \\ u^3(t) = 3\,\sin(0.1\,t + 1),\; u^4(t) = \log(t + 0.1), \\ u^5(t) = 0.02\,t,\; u^6(t) = 0.5\,(t - 25). \end{cases}$$

The simulation result is shown in Fig. 6. The onset of any changes in the network topology is indicated by vertical dotted lines. In Fig. 6, in order to convey clearly the incidences of the agents joining or leaving the network, for the time prior to joining and the time after departure, the fixed initial value or departure value is used, respectively. As the simulation shows in Fig. 6, the proposed filter *FOI-DCF* handles the changes in the topology well. When the change is just a switching in the network communication the filter is almost indifferent to the change. In the cases of agents joining or leaving the network, after a transient period, the filter presumes its close tracking of the input average. Here, we used $\beta = 1$ and $\epsilon = 0.01$.

D. Example 4 (Comparison between the performance of the proposed filters FOI-DCF and SOI-DCF when input signals are offset with a static value)

Consider the network depicted in Fig. 3 with the input signals listed below:

$$\begin{cases} u^1(t) = u(t) + 1,\; u^2(t) = u(t) - 1,\; u^3(t) = u(t) + 4, \\ u^4(t) = u(t) + 5,\; u^5(t) = u(t) + 10, \end{cases}$$

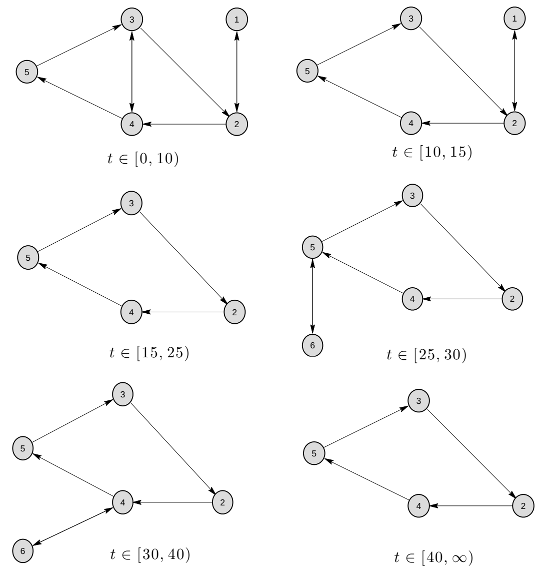


Fig. 5. Network of example 3

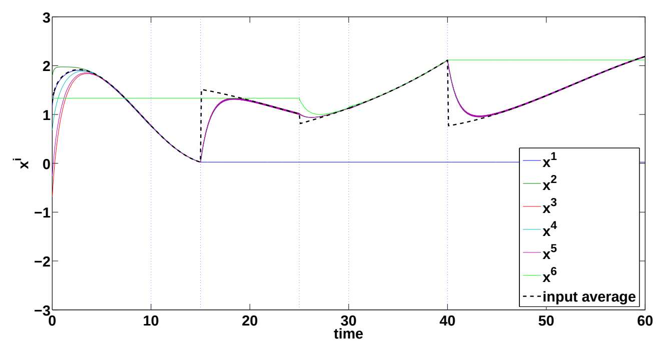


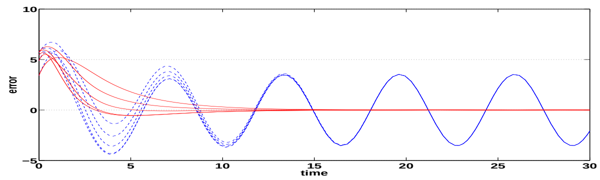
Fig. 6. Simulation result for example 3.

where $u(t) = 5\,\sin(t)$. These inputs differ from one and other by a static value. Figure 7 shows the time history of the error between the states x^i's and the dynamic input average. The dashed blue lines belong to the filter *FOI-DCF* and the solid red lines belong to *SOI-DCF*. Figure 7(a) is generated using $\epsilon = 1$, and Fig 7(b) is generated using $\epsilon = 0.01$. As expected the filter *SOI-DCF*, regardless of the value of ϵ, converges to the dynamic input average with no steady-state error. For a small ϵ, the performance of the filter *FOI-DCF* improves and almost matches the perfect performance of the filter *SOI-DCF*.

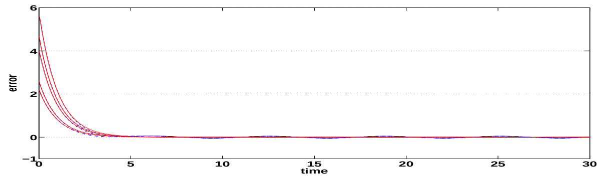
E. Example 5 (Comparison between the performance of the proposed filters FOI-DCF and (11) when agents do not use common β)

Consider the network depicted in Fig. 3 with the input signals listed in Example 2. Here, we compare the performance of the filters *FOI-DCF* and (11), when agents use the following β^i's

$$\beta^1 = 1.2,\;\; \beta^2 = 1,\;\; \beta^3 = 0.8,\;\; \beta^4 = 0.5,\;\; \beta^5 = 0.2.$$

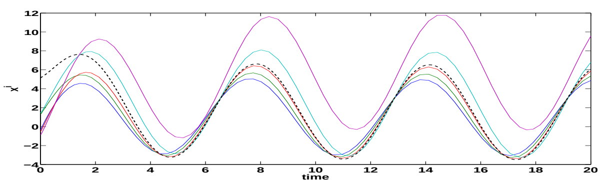


(a) $\epsilon = 1$ and $\beta = 1$

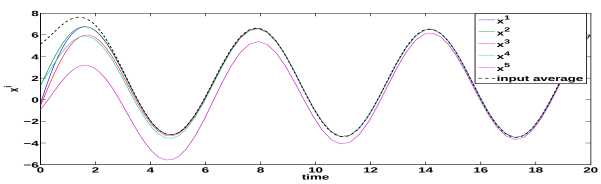


(b) $\epsilon = 0.01$ and $\beta = 1$

Fig. 7. Simulation results for example 4.



(a) The proposed filters *FOI-DCF*



(b) the proposed filter (11)

Fig. 8. Simulation results for example 5.

Simulation result, depicted in Fig. 8(a), indicate that the filter *FOI-DCF* does not approach to the right dynamic input average, as it is not robust with respect to β. However, the filter (11), which is designed to work for different values of β at each agent, converges to the input average. For the filter (11), as expected, the convergence rate at each agent is defined by its corresponding β^i; the larger the β^i; the faster the convergence is. Here we used $\epsilon = 0.001$.

F. Example 6 (performance of the proposed filters FOI-DCF with respect to discontinuous inputs)

Consider the network depicted in Fig. 3. The followings are the inputs at each agent

$$\begin{cases} u^1(t) = 3\,u(t)\,\cos(0.2\,t),\; u^2(t) = u(t)\tanh(0.2\,t), \\ u^3(t) = 3\,u(t)\,\sin(0.1\,t+1),\; u^4(t) = u(t)\,\log(t+0.1), \\ u^5(t) = 0.02\,u(t)\,t. \end{cases}$$

where $u(t) = \sum_{i=0}^{\infty}((-1)^i H(t-20\,i))$, in which H is the step function, i.e.,

$$H(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0. \end{cases}$$

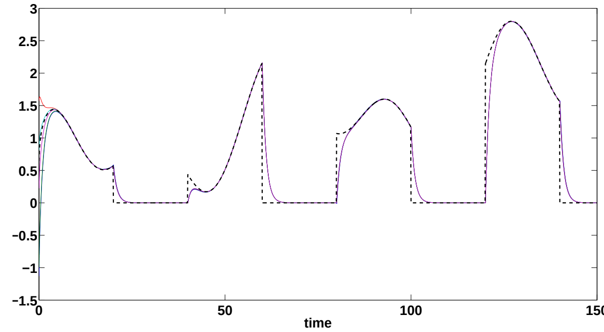


Fig. 9. Simulation result for example 6.

The simulation result obtained using the proposed filter *FOI-DCF* is shown in Fig. 9. Here we used $\epsilon = 0.01$. As shown in the figure, the proposed filter *FOI-DCF* achieves a close tracking despite the discontinuity in the input signals as long as dwell times are sufficiently large.

VI. Conclusions

We have proposed two continuous-time dynamic average consensus filters for networks with balanced and weakly-connected directed graph topology. The proposed filters have a two-time scale structure and do not require model information on the dynamic inputs. Using singular perturbation analysis, we have shown that the proposed filters reach an $O(\epsilon)$-neighborhood of the dynamic input average in finite time irrespectively of the initial condition. Simulation results show that the filters are robust to switching network topologies, as long as the network stays balanced and weakly-connected, and there is a large enough dwelling time between each switchings. Future work will be devoted to rigorously characterizing the convergence properties of the proposed dynamic average consensus algorithms over networks with switching topology.

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APPENDIX

For the sake of completeness, here we give a short review of Singularly Perturbed dynamical system theory and its terminology along with the theorem that we will employ to derive our main result on dynamic average consensus algorithms (see [15] for more details).

Consider the following system:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{z}, \epsilon), \qquad \mathbf{x}(t_0) = \boldsymbol{\eta}(\epsilon), \tag{12a}$$

$$\epsilon\, \dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{x}, \mathbf{z}, \epsilon), \qquad \mathbf{z}(t_0) = \boldsymbol{\zeta}(\epsilon). \tag{12b}$$

The use of a small constant $\epsilon > 0$ induces two time scales in the system, resulting into a fast and a slow dynamics. The analysis of such systems can be achieved with the aid of Singular Perturbation Theory. Singular Perturbation Theory establishes rigorous conditions under which the behavior of the system follows that of the limiting system when ϵ goes to 0. We assume that $\mathbf{f}$ and $\mathbf{g}$ are continuously differentiable in their arguments for $(t, \mathbf{x}, \mathbf{z}, \epsilon) \in [0, \infty) \times D_x \times D_z \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^m$ are open connected sets. When we set $\epsilon = 0$ in (12), the dimension of the state equation reduces from $n + m$ to n because the differential equation (12b) degenerates into the algebraic equation

$$\mathbf{0} = \mathbf{g}(t, \mathbf{x}, \mathbf{z}, 0). \tag{13}$$

We say that the model (12) is in standard form if (13) has $k \geq 1$ isolated real roots

$$\mathbf{z}_i = \mathbf{h}_i(t, \mathbf{x}), \quad i = 1, \cdots, k, \tag{14}$$

for each $(t, \mathbf{x}) \in [0, \infty) \times D_x$. This assumption assures that a well-defined $n-$dimensional *reduced model* (slow dynamics) will correspond to each root of (13). To obtain the ith reduced model, we substitute (14) into (12a), at $\epsilon = 0$, to obtain

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{h}(t, \mathbf{x}), 0), \tag{15}$$

where we have dropped the subscript i from $\mathbf{h}$. The *boundary-layer* system (fast dynamics) is

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{g}(t, \mathbf{x}, \mathbf{z}, 0), \qquad \tau = \frac{t}{\epsilon}, \tag{16}$$

where $\mathbf{x}$ and t are treated as fixed parameters. In the analysis, it is common to perform the change of variables $\mathbf{y} = \mathbf{z} - \mathbf{h}(t, \mathbf{x})$ that shifts the quasi-steady state of $\mathbf{z}$ to the origin.

Theorem A.1 ([15]): Consider the Singular Perturbation problem of (12) and let $\mathbf{z} = \mathbf{h}(t, \mathbf{x})$ be an isolated root of (13). Assume that, for all

$$[t, \mathbf{x}, \mathbf{z} - \mathbf{h}(t, \mathbf{x}), \epsilon] \in [0, \infty) \times D_x \times D_y \times [0, \epsilon_0],$$

where $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^m$ are domains which contain their respective origins, the following conditions hold:

- On any compact subset of $D_x \times D_y$, the functions $\mathbf{f}$, $\mathbf{g}$, their partial derivatives with respect to $(\mathbf{x}, \mathbf{z}, \epsilon)$, and the first partial derivative of $\mathbf{g}$ with respect to t are continuous and bounded; the functions $\mathbf{h}(t, \mathbf{x})$ and $[\partial \mathbf{g}(t, \mathbf{x}, \mathbf{z}, 0)/\partial \mathbf{z}]$ have bounded first partial derivatives with respect to their arguments; the function $[\partial \mathbf{f}(t, \mathbf{x}, \mathbf{h}(t, \mathbf{x}), 0)/\partial \mathbf{x}]$ is Lipschitz in $\mathbf{x}$, uniformly in t; and the initial conditions $\boldsymbol{\eta}(\epsilon)$ and $\boldsymbol{\zeta}(\epsilon)$ are smooth functions of ϵ;
- the origin is an exponentially stable equilibrium point of the reduced system (15); that is, there is a Lyapunov function $V(t, \mathbf{x})$ that satisfies the conditions of [15, Theorem 4.9] for (15) for $(t, \mathbf{x}) \in [0, \infty) \times D_x$. In other words, we have
$$W_1(\mathbf{x}) \leq V(t, \mathbf{x}) \leq W_2(\mathbf{x}),$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{h}(t, \mathbf{x}), 0) \leq -W_3(\mathbf{x}),$$
where W_1, W_2, and W_3 are continuous positive definite functions on D_x, such that $\{W_1(\mathbf{x}) \leq c\}$ is a compact subset of D_x;
- the origin is an exponentially stable equilibrium of the boundary layer system, uniformly in $(t, \mathbf{x})$. Let $\mathcal{R}_y \subset D_y$ be the region of attraction of
$$\frac{d\mathbf{y}}{d\tau} = \mathbf{g}(0, \boldsymbol{\eta}(0), \mathbf{y} + \mathbf{h}(0, \boldsymbol{\eta}(0)), 0),$$
$$\mathbf{y}(0) = \boldsymbol{\zeta}(0) - \mathbf{h}(0, \boldsymbol{\eta}(0)),$$
and Ω_y be a compact subset of $\mathcal{R}_y$.

Then, for each compact set $\Omega_x \subset \{W_2(\mathbf{x}) \leq \rho c, 0 < \rho < 1\}$ there is a positive constant $\epsilon^\star$ such that for all $t_0 \geq 0$, $\boldsymbol{\eta}_0 \in \Omega_x$, $\boldsymbol{\zeta}_0 - \mathbf{h}(t_0, \boldsymbol{\eta}_0) \in \Omega_y$, and $0 < \epsilon < \epsilon^\star$, the singularly perturbed system (12) has a unique solution $\mathbf{x}(t, \epsilon)$, $\mathbf{z}(t, \epsilon)$ on $[t_0, \infty)$, and

$$\mathbf{x}(t, \epsilon) - \bar{\mathbf{x}}(t) = O(\epsilon), \tag{17}$$

$$\mathbf{z}(t, \epsilon) - \mathbf{h}(t, \bar{\mathbf{x}}(t)) - \hat{\mathbf{y}}(t/\epsilon) = O(\epsilon) \tag{18}$$

hold uniformly for $t \in [t_0, \infty)$, where $\bar{\mathbf{x}}(t)$ and $\hat{\mathbf{y}}(\tau)$ are the solutions of the reduced and boundary layer problems. Moreover, given any $t_b > t_0$, there is $\epsilon^{\star\star} \leq \epsilon^\star$ such that

$$\mathbf{z}(t, \epsilon) - \mathbf{h}(t, \bar{\mathbf{x}}(t)) = O(\epsilon) \tag{19}$$

holds uniformly for $t \in [t_b, \infty)$ whenever $\epsilon < \epsilon^{\star\star}$.