Exact Sizing of Battery Capacity for Photovoltaic Systems

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Abstract

In this paper, we study battery sizing for grid-connected photovoltaic (PV) systems. In our setting, PV generated electricity is used to supply the demand from loads: on the one hand, if there is surplus PV generation, it is stored in a battery (as long as the battery is not fully charged), which has a fixed maximum charging/discharging rate; on the other hand, if the PV generation and battery discharging cannot meet the demand, electricity is purchased from the grid. Our objective is to choose an appropriate battery size while minimizing the electricity purchase cost from the grid. More specifically, we want to find a unique critical value (denoted as $E^{c}_{\max}$) of the battery size such that the cost of electricity purchase remains the same if the battery size is larger than or equal to $E^{c}_{\max}$, and the cost is strictly larger otherwise. We propose an upper bound on $E^{c}_{\max}$, and show that the upper bound is achievable for certain scenarios. For the case with ideal PV generation and constant loads, we characterize the exact value of $E^{c}_{\max}$, and also show how the storage size changes as the constant load changes; these results are validated via simulations.

Keywords:
PV, Grid, Battery, Optimization

1. Introduction

Installations of solar photovoltaic (PV) systems have been growing at a rapid pace in recent years due to the advantages of PV such as modest environmental impacts (clean energy), avoidance of fuel price risks, coincidence with peak electrical demand, and the ability to deploy PV at the point of use. In 2010, approximately 17,500 megawatts (MWs) of PV were installed globally, up from approximately 7,500 MWs in 2009, consisting primarily of grid-connected applications [1]. Since PV generation tends to fluctuate due to cloud cover and the daily solar cycle, energy storage devices, e.g., batteries, ultracapacitors, and compressed air, can be used to smooth out the fluctuation of the PV output fed into electric grids ("capacity firming") [2], discharge and augment the PV output during times of peak energy usage ("peak shaving") [3], or store energy for nighttime use, for example in zero-energy buildings.

In this paper, we study battery sizing for grid-connected PV systems to store energy for nighttime use. Our setting is shown in Fig. 1. PV generated electricity is used to supply loads: on the one hand, if there is surplus PV generation, it is stored in a battery for later use or dumped (if the battery is fully charged); on the other hand, if the PV generation and battery discharging cannot meet the demand, electricity is purchased from the grid. The battery has a fixed maximum charging/discharging rate. Our objective is to choose an appropriate battery size while minimizing the electricity purchase cost from the grid. We show that there is a unique critical value (denoted as $E^{\rm c}_{\max}$, refer to Problem 1) of the battery capacity (under fixed maximum charging and discharging rates) such that the cost of electricity purchase remains the same if the battery size is larger than or equal to $E^{\rm c}_{\max}$, and the cost is strictly larger otherwise. We first propose an upper bound on $E^{\rm c}_{\max}$ given the PV generation, loads, and the time period for minimizing the costs, and show that the upper bound becomes exact for certain scenarios. For the case of idealized PV generation (roughly, it refers to PV output on clear days) and constant loads, we analytically characterize the exact value of $E^{\rm c}_{\max}$, which is consistent with the critical value obtained via simulations.

The storage sizing problem has been studied for both off-grid and grid-connected applications. For example, the IEEE standard [4] provides sizing recommendations for lead-acid batteries in stand-alone PV systems. In [5], the solar panel size and the battery size have been selected via simulations to optimize the operation of a stand-alone PV system. If the PV system is grid-connected, batteries can reduce the fluctuation of PV output or provide economic benefits such as demand charge reduction, capacity firming, and power arbitrage. The work in [6] analyzes the relation between available battery capacity and output smoothing, and estimates the required battery capacity using simulations. In addition, the battery sizing problem has been studied for wind power applications [7, 8, 9] and hybrid wind/solar power applications [10, 11, 12]. Most previous work completely relies on trial and error approaches to calculate the storage size. Only very limited work has contributed to the theoretical analysis of storage sizing, such as [13, 14, 15]. In [13], discrete Fourier transforms are used to decompose the required balancing power into different time-varying periodic components, each of which can be used to quantify the physical maximum energy storage requirement. In [14], the storage sizing problem is cast as an infinite horizon stochastic optimization problem to minimize the long-term average cost of electric bills in the presence of dynamic pricing as well as investment in storage. In [15], we cast the storage sizing problem as a finite horizon deterministic optimization problem to minimize the cost associated with the net power purchase from the electric grid and the battery capacity loss due to aging while satisfying the load and reducing peak loads. Lower and upper bounds on the battery size are proposed that facilitate the efficient calculation of its value. The contribution of this work is the following: exact values of battery size for the special case of ideal PV generation and

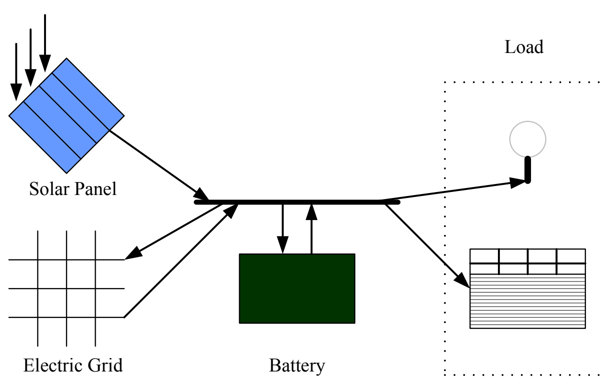


Figure 1: Grid-connected PV system with battery storage and loads.

constant loads are characterized; in contrast, in [15], only lower and upper bounds are obtained. In addition, the setting in this work is different from that of [14] in that a finite horizon deterministic optimization is formulated here. These results can be generalized to more practical PV generation and dynamic loads (as discussed in Remark 10).

We acknowledge that our analysis does not apply to the typical scenario of "net-metered" systems[1], where feed-in of energy to the grid is remunerated at the same rate as purchase of energy from the grid. Consequently, the grid itself acts as a storage system for the PV system (and $E^{\mathrm{c}}_{\max}$ becomes 0). However, from a grid operator standpoint it would be most desirable if PV system could just serve the local load and not export to the grid. This motivates our choice of no revenue for dumping power to the grid. Our scenario also has analogues at the level of a balancing area by avoiding curtailment or intra-hour energy export. For load balancing, in a balancing area (typically a utility grid) steady-state conditions are set every hour. This means that the power imports are constant over the hour. The balancing authority then has to balance local generation with demand such that the steady state will be preserved. This also corresponds to avoiding "outflow" of energy from the balancing area. In a grid with very high renewable penetration, there may be more renewable production than load. In that case, the energy would be dumped or "curtailed". However, with demand response (e.g., loads with relatively flexible schedules) or battery storage, curtailment could be avoided.

The paper is organized as follows. In the next section, we introduce our setting, and formulate the battery sizing problem. An upper bound on $E^{\mathrm{c}}_{\max}$ is proposed in Section 3, and the exact value of $E^{\mathrm{c}}_{\max}$ is obtained for ideal PV generation and constant loads in Section 4. In Section 5, we validate the results via simulations. Finally, conclusions and future directions are given in Section 6.

2. Problem Formulation

In this section, we formulate the problem of determining the storage size for grid-connected PV system, as shown in Fig. 1. Solar panels are used to generate electricity, which can be used to supply loads, e.g., lights, air conditioners, microwaves in a residential setting. On the one hand, if there is surplus electricity, it can be stored in a battery, or dumped to the grid if the battery is fully charged. On the other hand, if there is not enough electricity to power the loads, electricity can be drawn from the electric grid. Before formalizing the battery sizing problem, we first introduce different components in our setting.

2.1. Photovoltaic Generation

We use the following equation to calculate the electricity generated from solar panels:

$$P_{\mathrm{pv}}(t) = \mathrm{GHI}(t) \times S \times \eta \,, \tag{1}$$

where GHI (Wm^{-2}) is the global horizontal irradiation at the location of solar panels, S (m^2) is the total area of solar panels, and η is the solar conversion efficiency of the PV cells. The PV generation model is a simplified version of the one used in [16] and does not account for PV panel temperature effects.

[1] Note that in [15], we study battery sizing for "net-metered" systems under more relaxed assumptions compared with this work.

2.2. *Electric Grid*

Electricity can be drawn from (or dumped to) the grid. We associate costs only with the electricity purchase from the grid, and assume that there is no benefit by dumping electricity to the grid. The motivation is that, from a grid operator standpoint, it would be most desirable if PV system could just serve the local load and not export to the grid. In a grid with very high renewable penetration, there may be more renewable production than load. In that case, the energy would have to be dumped (or curtailed).

We use $C_{\text{gp}}(t)$ (¢/kWh) to denote the electricity purchase rate, $P_{\text{gp}}(t)(W)$ to denote the electricity purchased from the grid at time t, and $P_{\text{gd}}(t)(W)$ to denote the surplus electricity dumped to the grid or curtailed at time t. For simplicity, we assume that $C_{\text{gp}}(t)$ is *time independent* and has the value C_{gp}. In other words, there is no difference between the electricity purchase rates at different time instants.

2.3. *Battery*

A battery has the following dynamic:

$$\frac{dE_B(t)}{dt} = P_B(t) , \tag{2}$$

where $E_B(t)(Wh)$ is the amount of electricity stored in the battery at time t, and $P_B(t)(W)$ is the charging/discharging rate (more specifically, $P_B(t) > 0$ if the battery is charging, and $P_B(t) < 0$ if the battery is discharging). We impose the following constraints on the battery:

i) At any time, the battery charge $E_B(t)$ should satisfy $E_{\text{Bmin}} \leq E_B(t) \leq E_{\text{Bmax}}$, where E_{Bmin} is the minimum battery charge, E_{Bmax} is the maximum battery charge, and[2] $0 < E_{\text{Bmin}} \leq E_{\text{Bmax}}$, and

ii) The battery charging/discharging rate should satisfy $P_{\text{Bmin}} \leq P_B(t) \leq P_{\text{Bmax}}$, where $P_{\text{Bmin}} < 0$, $-P_{\text{Bmin}}$ is the maximum battery discharging rate, and $P_{\text{Bmax}} > 0$ is the maximum battery charging rate.

For lead-acid batteries, more complicated models exist (e.g., a third order model is proposed in [17, 18]).

2.4. *Load*

$P_{\text{load}}(t)(W)$ denotes the load at time t. We do not make explicit assumptions on the load considered in Section 3 except that $P_{\text{load}}(t)$ is a (piecewise) continuous function. Loads could have a fixed schedule such as lights and TVs, or a relatively flexible schedule such as refrigerators and air conditioners. For example, air conditioners can be turned on and off with different schedules as long as the room temperature is within a comfortable range. In Section 4, we consider constant loads, i.e., $P_{\text{load}}(t)$ is independent of time t.

[2]Usually, E_{Bmin} is chosen to be larger than 0 to prevent fast battery aging. For detailed modeling of the aging process, refer to [3].

2.5. Minimization of Electricity Purchase Cost

With all the components introduced earlier, now we can formulate the following problem of minimizing the electricity purchase cost from the electric grid while guaranteeing that the demand from loads are satisfied:

$$
\begin{aligned}
\min_{P_B, P_{\mathrm{gp}}, P_{\mathrm{gd}}} & \int_{t_0}^{t_0+T} C_{\mathrm{gp}} P_{\mathrm{gp}}(\tau) d\tau \\
\text{s.t. } & P_{\mathrm{pv}}(t) + P_{\mathrm{gp}}(t) = P_{\mathrm{gd}}(t) + P_B(t) + P_{\mathrm{load}}(t) \,, \qquad (3) \\
& \frac{dE_B(t)}{dt} = P_B(t) \,, E_B(t_0) = E_{\mathrm{Bmin}} \,, \\
& E_{\mathrm{Bmin}} \leq E_B(t) \leq E_{\mathrm{Bmax}} \,, \\
& P_{\mathrm{Bmin}} \leq P_B(t) \leq P_{\mathrm{Bmax}} \,, \\
& P_{\mathrm{gp}}(t) \geq 0, P_{\mathrm{gd}}(t) \geq 0 \,, \qquad (4)
\end{aligned}
$$

where t_0 is the initial time, T is the time period considered for the cost minimization. Eq. (3) is the power balance requirement for any time $t \in [t_0, t_0+T]$; in other words, the supply of electricity (either from PV generation, grid purchase, or battery discharging) must meet the demand.

2.6. Battery Sizing

Based on Eq. (3), we obtain

$$
P_{\mathrm{gp}}(t) = P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t) + P_B(t) + P_{\mathrm{gd}}(t) \,.
$$

Then the optimization problem in Eq. (4) can be rewritten as

$$
\begin{aligned}
\min_{P_B, P_{\mathrm{gd}}} & \int_{t_0}^{t_0+T} C_{\mathrm{gp}} (P_{\mathrm{load}}(\tau) - P_{\mathrm{pv}}(\tau) + P_B(\tau) + P_{\mathrm{gd}}(\tau)) d\tau \\
\text{s.t.} & \frac{dE_B(t)}{dt} = P_B(t) \,, E_B(t_0) = E_{\mathrm{Bmin}} \,, \\
& E_{\mathrm{Bmin}} \leq E_B(t) \leq E_{\mathrm{Bmax}} \,, \\
& P_{\mathrm{Bmin}} \leq P_B(t) \leq P_{\mathrm{Bmax}} \,, \\
& P_{\mathrm{gd}}(t) \geq 0 \,.
\end{aligned}
$$

Now there are two independent variables $P_B(t)$ and $P_{\mathrm{gd}}(t)$. To minimize the total electricity purchase cost, we have the following key observations:

(A) If the battery is charging, i.e., $P_B(t) > 0$ and $E_B(t) < E_{\mathrm{Bmax}}$, then the charged electricity should only come from surplus PV generation;

(B) If the battery is discharging, i.e., $P_B(t) < 0$ and $E_B(t) > E_{\mathrm{Bmin}}$, then the discharged electricity should only be used to supply loads. In other words, the dumped electric power $P_{\mathrm{gd}}(t)$ should only come from surplus PV generation.

In observation (A), the battery can be charged by purchasing electricity from the grid at the current time and be used later on when the PV generated electricity is insufficient to meet demands. However, this incurs a cost at the current time, and the saving of costs by discharging later on is the same as the cost of charging (or could be less than the cost of charging if the discharged electricity gets dumped to the grid) because the electricity purchase rate is time independent. That is to say, there is no gain in terms of costs by operating the battery charging in this way. Therefore, we can restrict the battery charging to using only PV generated electric power. In observation (B), if the battery charge is dumped to the grid, potentially it could increase the total cost since extra electricity might need to be purchased from the grid to meet the demand. In summary, we have the following rule to operate the battery and dump electricity to the grid (i.e., restricting the set of possible control actions) without increasing the total cost.

Rule 1 The battery gets charged from the PV generation only when there is surplus PV generated electric power and the battery can still be charged, and gets discharged to supply the load only when the load cannot be met by PV generated electric power and the battery can still be discharged. PV generated electric power gets dumped to the grid only when there is surplus PV generated electric power other than supplying both the load and the battery charging.

With this operating rule, we can further eliminate the variable $P_{\mathrm{gd}}(t)$ and obtain another equivalent optimization problem. On the one hand, if $P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t) + P_B(t) < 0$, i.e., the electricity generated from PV is more than the electricity consumed by the load and charging the battery, we need to choose $P_{\mathrm{gd}}(t) > 0$ to make $P_{\mathrm{gp}}(t) = 0$ so that the cost is minimized; on the other hand, if $P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t) + P_B(t) > 0$, i.e., the electricity generated from PV and battery discharging is less than the electricity consumed by the load, we need to choose $P_{\mathrm{gd}}(t) = 0$ to minimize the electricity purchase costs, and we have $P_{\mathrm{gp}}(t) = P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t) + P_B(t)$. Therefore, $P_{\mathrm{gp}}(t)$ can be written as

$$P_{\mathrm{gp}}(t) = \max(0, P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t) + P_B(t)) ,$$

so that the integrand is minimized at each time.

Let

$$x(t) = E_B(t) - \frac{E_{\mathrm{Bmax}} + E_{\mathrm{Bmin}}}{2} ,$$

$u(t) = P_B(t)$, and

$$E_{\max} = \frac{E_{\mathrm{Bmax}} - E_{\mathrm{Bmin}}}{2} .$$

Note that $2E_{\max} = E_{\mathrm{Bmax}} - E_{\mathrm{Bmin}}$ is the net (usable) battery capacity, which is the maximum amount of electricity that can be stored in the battery. Then the optimization problem can be rewritten as

$$\begin{aligned} J = \min_u & \int_{t_0}^{t_0+T} C_{\mathrm{gp}} \max(0, P_{\mathrm{load}}(\tau) - P_{\mathrm{pv}}(\tau) + u(\tau)) d\tau \\ \text{s.t. } & \frac{dx(t)}{dt} = u(t) , \, x(t_0) = -E_{\max} , \\ & |x(t)| \le E_{\max} , \\ & P_{\mathrm{Bmin}} \le u(t) \le P_{\mathrm{Bmax}} . \end{aligned} \tag{5}$$

Now it is clear that only $u(t)$ (or equivalently, $P_B(t)$) is an independent variable. As argued previously, we can restrict $u(t)$ to satisfying Rule 1 without increasing the minimum cost J. We define the set of feasible controls (denoted as U_{feasible}) as controls that satisfy the constraint $P_{\mathrm{Bmin}} \le u(t) \le P_{\mathrm{Bmax}}$ and do not violate Rule 1.

If we *fix* the parameters t_0, T, P_{Bmin}, and P_{Bmax}, J is a function of $E_{\max}$, which is denoted as $J(E_{\max})$. If $E_{\max} = 0$, then $u(t) = 0$, and J reaches the largest value

$$J_{\max} = \int_{t_0}^{t_0+T} C_{\text{gp}} \max(0, P_{\text{load}}(\tau) - P_{\text{pv}}(\tau)) d\tau \ . \tag{6}$$

If we increase $E_{\max}$, intuitively J will decrease (though may not strictly decrease) because the battery can be utilized to store extra electricity generated from PV to be potentially used later on when the load exceeds the PV generation. This is justified in the following proposition.

Proposition 1 Given the optimization problem in Eq. (5), if $E^1_{\max} < E^2_{\max}$, then $J(E^1_{\max}) \geq J(E^2_{\max})$.

Proof Refer to Appendix A.

In other words, J is monotonically decreasing with respect to the parameter $E_{\max}$, and is lower bounded by 0. We are interested in finding the smallest value of $E_{\max}$ (denoted as $E^{\text{c}}_{\max}$) such that J remains the same for any $E_{\max} \geq E^{\text{c}}_{\max}$, and call it the battery sizing problem.

Problem 1 (**Battery Sizing**) Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax}, determine a critical value $E^{\text{c}}_{\max} \geq 0$ such that $\forall E_{\max} < E^{\text{c}}_{\max}$, $J(E_{\max}) > J(E^{\text{c}}_{\max})$, and $\forall E_{\max} \geq E^{\text{c}}_{\max}$, $J(E_{\max}) = J(E^{\text{c}}_{\max})$.

Remark 1 In the battery sizing problem, we fix the charging and discharging rate of the battery while varying the battery capacity. This is reasonable if the battery is charged with a fixed charger, which uses a constant charging voltage but can change the charging current within a certain limit. In practice, the charging and discharging rates could scale with $E_{\max}$, which results in challenging problems to solve and requires further study. ■

Note that the critical value $E^{\text{c}}_{\max}$ is unique as shown in the following proposition, which can be proved via contradiction.

Proposition 2 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax}, $E^{\text{c}}_{\max}$ is unique.

Remark 2 One idea to calculate the critical value $E^{\text{c}}_{\max}$ is that we first obtain an explicit expression for the function $J(E_{\max})$ by solving the optimization problem in Eq. (5) and then solve for $E^{\text{c}}_{\max}$ based on the function J. However, the optimization problem in Eq. (5) is difficult to solve due to the state constraint $|x(t)| \leq E_{\max}$ and the fact that it is hard to obtain analytical expressions for $P_{\text{load}}(t)$ and $P_{\text{pv}}(t)$ in reality. Even though it might be possible to find the optimal control using the minimum principle [19], it is still hard to get an explicit expression for the cost function J. Instead, in the next section, we first focus on bounding the critical value $E^{\text{c}}_{\max}$ in general, and then study the problem for specific scenarios in Section 4. ■

3. Upper Bound on $E^{\text{c}}_{\max}$

In this section, we first identify necessary assumptions to ensure a nonzero $E^{\text{c}}_{\max}$, then propose an upper bound on $E^{\text{c}}_{\max}$, and finally show that the upper bound is tight for certain scenarios.

Proposition 3 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax}, $E^{\text{c}}_{\max} = 0$ if any of the following conditions holds:

(i) $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \leq 0$,

(ii) $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \geq 0$,

(iii) $\forall t_1 \in S_1$, $\forall t_2 \in S_2$, $t_2 < t_1$, where

$$S_1 := \{t \in [t_0, t_0 + T] \mid P_{\text{pv}}(t) - P_{\text{load}}(t) > 0\}\ , \tag{7}$$

$$S_2 := \{t \in [t_0, t_0 + T] \mid P_{\text{load}}(t) - P_{\text{pv}}(t) > 0\}\ . \tag{8}$$

Proof Refer to Appendix B.

Remark 3 The intuition of condition (i) in Proposition 3 is that if $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \leq 0$, no extra electricity is generated from PV and can be stored in the battery to strictly reduce the cost. The intuition of condition (ii) in Proposition 3 is that if $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \geq 0$, the electricity generated from PV alone is enough to satisfy the load all the time, and extra electricity can be simply dumped to the grid. Note that $J_{\max} = 0$ for this case. As defined in condition (iii), $S_1 \cap S_2 = \emptyset$ because it is impossible to have both $P_{\text{pv}}(t) - P_{\text{load}}(t) > 0$ and $P_{\text{load}}(t) - P_{\text{pv}}(t) > 0$ at the same time for any time t. ■

Based on the result in Proposition 3, we impose the following assumption on Problem 1 to make use of the battery.

Assumption 1 There exists t_1 and t_2 for $t_1, t_2 \in [t_0, t_0+T]$, such that $t_1 < t_2$, $P_{\text{pv}}(t_1)-P_{\text{load}}(t_1) > 0$ and $P_{\text{pv}}(t_2) - P_{\text{load}}(t_2) < 0$.

Remark 4 $P_{\text{pv}}(t_1) - P_{\text{load}}(t_1) > 0$ implies that at time t_1 there is surplus electric power available from PV. $P_{\text{pv}}(t_2) - P_{\text{load}}(t_2) < 0$ implies that at time t_2 the electric power from PV is not sufficient for the load. If $t_1 < t_2$, the electricity stored in the battery at time t_1 can be discharged to supply the load at time t_2 to strictly reduce the cost. ■

Proposition 4 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax} under Assumption 1, $0 < E^{\text{c}}_{\max} \leq \frac{\min(A,B)}{2}$, where

$$A = \int_{t_0}^{t_0+T} \min(P_{\text{Bmax}}, \max(0, P_{\text{pv}}(t) - P_{\text{load}}(t)))dt\ , \tag{9}$$

and

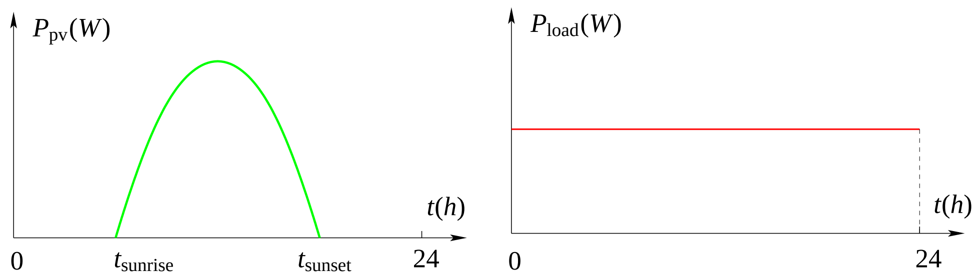
$$B = \int_{t_0}^{t_0+T} \min(-P_{\text{Bmin}}, \max(0, P_{\text{load}}(t) - P_{\text{pv}}(t)))dt\ . \tag{10}$$

Proof Refer to Appendix C.

Remark 5 Note that if $\forall t \in [t_0, t_0+T]$ we have $P_{\text{pv}}(t)-P_{\text{load}}(t) \leq 0$, then $A = 0$ following Eq. (9); therefore, the upper bound for $E^{\text{c}}_{\max}$ in Proposition 4 becomes 0, which implies that $E^{\text{c}}_{\max} = 0$. If $\forall t \in [t_0, t_0 + T]$ we have $P_{\text{pv}}(t) - P_{\text{load}}(t) \geq 0$, then $B = 0$ following Eq. (10); therefore, the upper bound for $E^{\text{c}}_{\max}$ in Proposition 4 becomes 0, which implies that $E^{\text{c}}_{\max} = 0$. Both results are consistent with the results in Proposition 3. ■

Proposition 5 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax} under Assumption 1, if $\forall t_1 \in S_1$, $\forall t_2 \in S_2$, $t_1 < t_2$, then $E^{\text{c}}_{\max} = \frac{\min(A,B)}{2}$, where S_1 (or S_2, A, B, respectively) is defined in Eq. (7) (or (8), (9), (10), respectively). In addition, if $E_{\max}$ is chosen to be $\frac{\min(A,B)}{2}$, $x(t_0 + T) = -E_{\max}$ (i.e., no battery charge left at time $t_0 + T$).

Proof Refer to Appendix D.



(a) Ideal PV generation for a clear day. (b) Constant load.

Figure 2: Ideal PV generation and constant load.

4. Ideal PV Generation and Constant Load

In this section, we study how to obtain the critical value for the scenario in which the PV generation is ideal and the load is constant. Ideal PV generation occurs on clear days; for a typical south-facing PV array on a clear day, the PV output is zero before about sunrise, rises continuously and monotonically to its maximum around solar noon, then decreases continuously and monotonically to zero around sunset, as shown in Fig. 2(a). In other words, there is essentially no short time fluctuation (at the scale of seconds to minutes) due to atmospheric effects such as clouds or precipitation. By constant load, we mean $P_{\text{load}}(t)$ is a constant for $t \in [t_0, t_0 + T]$. A typical constant load is plotted in Fig. 2(b). To further simplify the problem, we assume that t_0 is 0000 h Local Standard Time (LST), and $T = t_0 + k \times 24(h)$ where k is a nonnegative integer, i.e., T is a duration of multiple days. Fig. 2 plots the ideal PV generation and the constant load for $T = 24(h)$. Now we summarize these conditions in the following assumption.

Assumption 2 The initial time t_0 is 0000 h LST, $T = k \times 24(h)$ where k is a positive integer, $P_{\text{pv}}(t)$ is periodic on a timescale of 24 hours, and satisfies the following property for $t \in [0, 24(h)]$: there exist three time instants $0 < t_{\text{sunrise}} < t_{\text{max}} < t_{\text{sunset}} < 24(h)$ such that

- $P_{\text{pv}}(t) = 0$ for $t \in [0, t_{\text{sunrise}}] \cup [t_{\text{sunset}}, 24(h)]$;
- $P_{\text{pv}}(t)$ is continuous and strictly increasing for $t \in [t_{\text{sunrise}}, t_{\text{max}}]$;
- $P_{\text{pv}}(t)$ achieves its maximum $P_{\text{pv}}^{\text{max}}$ at t_{max};
- $P_{\text{pv}}(t)$ is continuous and strictly decreasing for $t \in [t_{\text{max}}, t_{\text{sunset}}]$,

and $P_{\text{load}}(t) = P_{\text{load}}$ for $t \in [t_0, t_0 + T]$, where P_{load} is a constant satisfying $0 < P_{\text{load}} < P_{\text{pv}}^{\text{max}}$.

It can be verified that Assumption 2 implies Assumption 1.

Proposition 6 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax} under Assumption 2 and $T = 24(h)$, $E_{\text{max}}^{\text{c}} = \frac{\min(A_1, B_1)}{2}$, where

$$A_1 = \int_{t_1}^{t_2} \min(P_{\text{Bmax}}, \max(0, P_{\text{pv}}(t) - P_{\text{load}}))dt ,$$

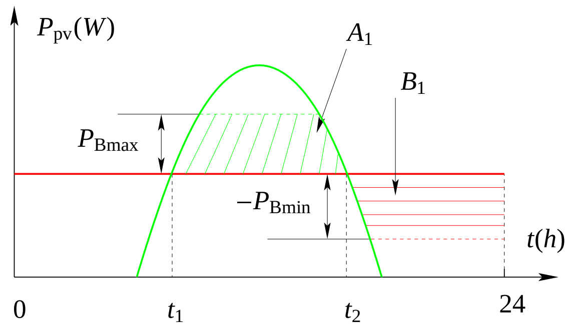


Figure 3: PV generation and load, where $P_{\rm Bmax}$ (or $-P_{\rm Bmin}$) is the maximum charging (or discharging) rate, and A_1, B_1, t_1, t_2 are defined in Proposition 6.

and

$$B_1 = \int_{t_2}^{24} \min(-P_{\rm Bmin}, \max(0, P_{\rm load} - P_{\rm pv}(t)))dt \ ,$$

in which $t_1 < t_2$ and $P_{\rm pv}(t_1) = P_{\rm pv}(t_2) = P_{\rm load}$.

Proof Refer to Appendix E.

Remark 6 A_1 and B_1 in Proposition 6 are shown in Fig. 3. In words, A_1 is the amount of extra PV generated electricity that can be stored in a battery, and B_1 is the amount of electricity that is necessary to supply the load *and* can be provided by battery discharging. Note that t_1 and t_2 depend on the value of $P_{\rm load}$. To eliminate this dependency, we can rewrite A_1 as

$$A_1 = \int_{0}^{24} \min(P_{\rm Bmax}, \max(0, P_{\rm pv}(t) - P_{\rm load}))dt \ , \tag{11}$$

and rewrite B_1 as

$$B_1 = \int_{t_{\rm max}}^{24} \min(-P_{\rm Bmin}, \max(0, P_{\rm load} - P_{\rm pv}(t)))dt \ , \tag{12}$$

where $t_{\rm max}$ is defined in Assumption 2. ■

Remark 7 If the PV generation is not ideal, i.e., there are fluctuations due to clouds or precipitation, the $E^{\rm c}_{\rm max}$ value in Proposition 6 based on ideal PV generation naturally serves as an upper bound on $E^{\rm c}_{\rm max}$ for the case with the non-ideal PV generation. Similarly, if the load varies with time but is bounded by a constant $P_{\rm load}$, the $E^{\rm c}_{\rm max}$ values in Proposition 6 based on the constant load $P_{\rm load}$ naturally serves as an upper bound on $E^{\rm c}_{\rm max}$ for the case with the time varying load. ■

Now we examine how $E^{\rm c}_{\rm max}$ changes as $P_{\rm load}$ varies from 0 to $P^{\rm max}_{\rm pv}$.

Proposition 7 Given the optimization problem in Eq. (5) with fixed $t_0, T, P_{\rm Bmin}$ and $P_{\rm Bmax}$ under Assumption 2 and $T = 24(h)$, then

a) there exists a unique critical value of $P_{\rm load} \in (0, P^{\rm max}_{\rm pv})$ (denoted as $P^c_{\rm load}$) such that $E^{\rm c}_{\rm max}$ achieves its maximum;

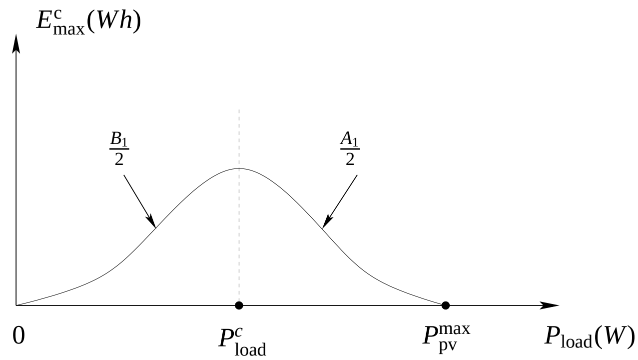


Figure 4: E^{c}_{max} as a function of P_{load} for $0 \le P_{load} \le P^{max}_{pv}$.

b) if P_{load} increases from 0 to P^{c}_{load}, E^{c}_{max} increases continuously and monotonically from 0 to its maximum;

c) if P_{load} increases from P^{c}_{load} to P^{max}_{pv}, E^{c}_{max} decreases continuously and monotonically from its maximum to 0.

Proof Refer to Appendix F.

Remark 8 A typical plot of E^{c}_{max} as a function of P_{load} is given in Fig. 4. Note that the slopes at 0 and P^{max}_{pv} are both 0, which can be derived from the expressions of $\frac{dA_1}{dP_{load}}$ and $\frac{dB_1}{dP_{load}}$. The result has the implication that there is a (finite) unique battery capacity that minimizes the grid electricity purchase cost for any $P_{load} > 0$. Fig. 10(a) verifies the plot via simulations. ■

Now we focus on the case with multiple days.

Proposition 8 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax} under Assumption 2 and $T = k \times 24(h)$ where $k > 1$ is a positive integer, $E^{c}_{max} = \frac{\min(A_2,B_2)}{2}$, where

$$A_2 = \int_{t_1}^{t_2} \min(P_{Bmax}, \max(0, P_{pv}(t) - P_{load}))dt \ ,$$

and

$$B_2 = \int_{t_2}^{t_3} \min(-P_{Bmin}, \max(0, P_{load} - P_{pv}(t)))dt \ ,$$

in which $t_i \in T_{crossing} := \{t \in [0, k \times 24] \mid P_{pv}(t) = P_{load}\}$, t_1, t_2, t_3 are the smallest three time instants in $T_{crossing}$ and satisfy $t_1 < t_2 < t_3$.

Proof Refer to Appendix G.

Remark 9 A_2 and B_2 in Proposition 8 are shown in Fig. 5. In words, A_2 is the amount of extra PV generated electricity that can be stored in a battery in the time interval $[t_1, t_3]$, and B_2 is the amount of electricity that is necessary to supply the load *and* can be provided by battery discharging in the time interval $[t_1, t_3]$. Note that t_1, t_2, and t_3 depend on the value of P_{load}. To eliminate this dependency, we can rewrite A_2 as

$$A_2 = \int_{0}^{24} \min(P_{Bmax}, \max(0, P_{pv}(t) - P_{load}))dt \ , \tag{13}$$

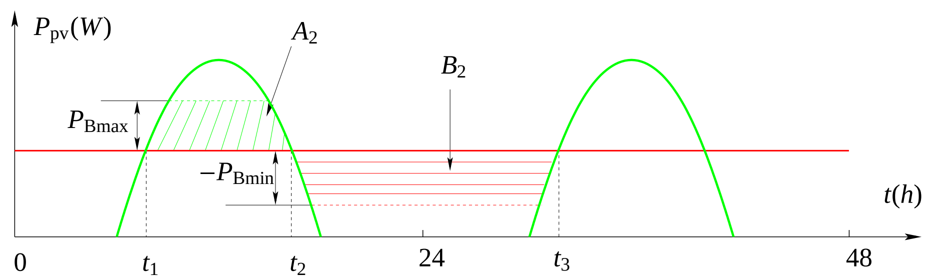


Figure 5: PV generation and load (two days), where P_{Bmax} (or $-P_{Bmin}$) is the maximum charging (or discharging) rate, and A_2, B_2, t_1, t_2, t_3 are defined in Proposition 8.

and rewrite B_2 as

$$B_2 = \int_{t_{max}}^{t_{max}+24} \min(-P_{Bmin}, \max(0, P_{load} - P_{pv}(t)))dt\ , \tag{14}$$

where t_{max} is defined in Assumption 2. ■

It can be verified that the following result on how E^c_{max} changes holds based on an analysis similar to the one in Proposition 7 using Eqs. (13) and (14), and Figs. 10(b) and (c) verify the trend via simulations.

Proposition 9 Given the optimization problem in Eq. (5) with fixed t_0, T, P_{Bmin} and P_{Bmax} under Assumption 2 and $T = k \times 24(h)$ where $k > 1$ is a positive integer, then

a) there exists a unique critical value of $P_{load} \in (0, P^{max}_{pv})$ (denoted as P^c_{load}) such that E^c_{max} achieves its maximum;

b) if P_{load} increases from 0 to P^c_{load}, E^c_{max} increases continuously and monotonically from 0 to its maximum;

c) if P_{load} increases from P^c_{load} to P^{max}_{pv}, E^c_{max} decreases continuously and monotonically from its maximum to 0.

Remark 10 Note that Assumption 2 can be relaxed. Given $T = 24$, if $P_{load}(t)$ and $P_{pv}(t)$ are piecewise continuous functions, and intersect at two time instants t_1, t_2, in addition S_1 (as defined in Eq. (7)) is the same as the open interval (t_1, t_2), then the result in Proposition 6 also holds, which can be proved similarly based on the argument in Proposition 6. Besides these conditions, if $P_{load}(t)$ and $P_{pv}(t)$ are periodic with period 24 hours, then the result in Proposition 8 also holds. However, with these relaxed conditions, the results in Propositions 7 and 9 do not hold any more since the load might not be constant. ■

5. Simulations

In this section, we verify the results in Sections 3 and 4 via simulations. The parameters used in Section 2 are chosen based on a typical residential home setting.

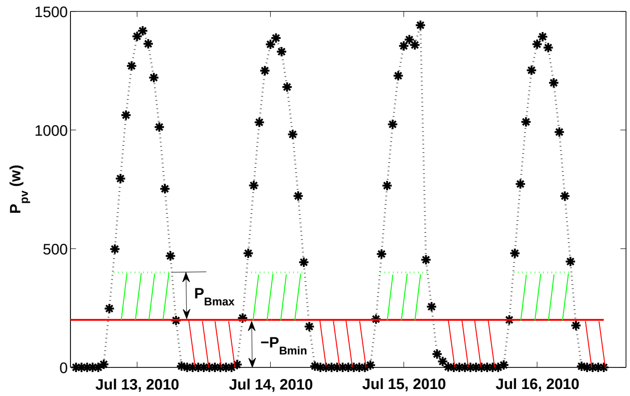


Figure 6: PV output for July 13 - 16, 2010, at La Jolla, California. For reference, a constant load 200 W is also shown (solid red line) along with $P_{\mathrm{Bmax}} = -P_{\mathrm{Bmin}} = 200\ W$. The green area corresponds to the amount of electricity that can be potentially charged to a battery, while the red area corresponds to the amount of electricity that can potentially be provided by discharging the battery.

The GHI data is the measured GHI between July 13 and July 16, 2010 at La Jolla, California. In our simulations, we use $\eta = 0.15$, and $S = 10m^2$. Thus $P_{\mathrm{pv}}(t) = 1.5 \times \mathrm{GHI}(t)(W)$. We choose t_0 as 0000 h LST on Jul 13, 2010, and the hourly PV output is given in Fig. 6 for the following four days starting from t_0. Except the small variation on Jul 15, 2010 and being not exactly periodic for every 24 hours, the PV generation roughly satisfies Assumption 2, which implies that Assumption 1 holds. Note that $0 \leq P_{\mathrm{pv}}(t) < 1500\ W$ for $t \in [t_0, t_0 + 96]$.

The electricity purchase rate C_{gp} is chosen to be 7.8¢/kWh, which is the semipeak rate for the summer season proposed by SDG&E (San Diego Gas & Electric) [20]. For the battery, we choose $E_{\mathrm{Bmin}} = 0.4 \times E_{\mathrm{Bmax}}$, and then

$$E_{\max} = \frac{E_{\mathrm{Bmax}} - E_{\mathrm{Bmin}}}{2} = 0.3 \times E_{\mathrm{Bmax}} \ .$$

The maximum charging rate is chosen to be $P_{\mathrm{Bmax}} = 200\ W$, and $P_{\mathrm{Bmin}} = -P_{\mathrm{Bmax}}$. Note that the battery dynamic is characterized by a continuous ordinary differential equation. To run simulations, we use one hour as the sampling interval, and discretize Eq. (2) as

$$E_B(k+1) = E_B(k) + P_B(k) \ .$$

5.1. Dynamic Loads

We first examine the upper bound in Proposition 4 using dynamic loads. The load profile for one day is given in Fig. 7, which resembles the residential load profile in[3] Fig. 8(b) in [16]. Note that one load peak appears in the early morning, and the other in the evening. For multiple day simulations, the load is periodic based on the load profile in Fig. 7. We study how the cost function J of the optimization problem in Eq. (5) changes as a function of $E_{\max}$ by increasing the battery capacity $E_{\max}$ from 0 to 1500 Wh with the step size 10 Wh. We solve the optimization

[3]However, simulations in [16] start at 7AM so Fig. 7 is a shifted version of the load profile in Fig. 8(b) in [16].

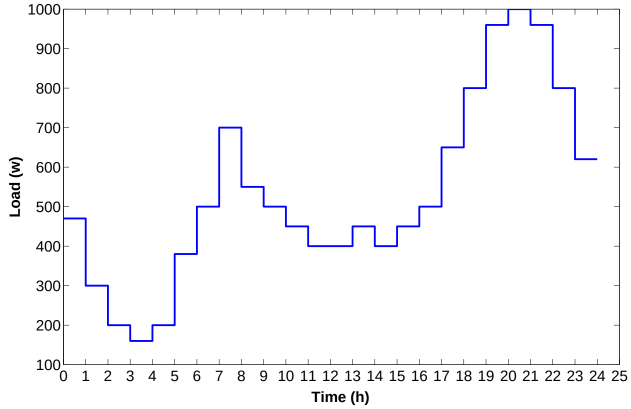


Figure 7: A typical residential load profile.

problem in Eq. (5) via linear programming using the CPlex sover [21]. If $T = 24(h)$ (or $T = 48(h)$, $T = 96(h)$, respectively), the plot of the minimum costs versus $E_{\max}$ is given in Fig. 8(a) (or (b), (c), respectively). The plots confirm the result in Proposition 1, i.e., the minimum cost is a decreasing function of $E_{\max}$, and also show the existence of the unique $E^{c}_{\max}$. If $T = 24(h)$, $E^{c}_{\max}$ is 700 Wh, which can be identified from Fig. 8(a). The upper bound in Proposition 4 is calculated to be 900 Wh. Similarly, if $T = 48(h)$, $E^{c}_{\max}$ is 900 Wh while the upper bound in Proposition 4 is calculated to be 1800 Wh; if $T = 96(h)$, $E^{c}_{\max}$ is 900 Wh while the upper bound in Proposition 4 is calculated to be 3402 Wh. The upper bound holds for these three cases though the difference between the upper bound and $E^{c}_{\max}$ increases when T increases. This is due to the fact that during multiple days battery can be repeatedly charged and discharged; however, this fact is not taken into account in the upper bound in Proposition 4. Since the load profile roughly satisfies the conditions imposed in Remark 10, we can calculate the theoretical value for $E^{c}_{\max}$ based on the results in Propositions 6 and 8 even for this dynamic load. If $T = 24(h)$, the theoretical value is 690 Wh which is obtained from Proposition 6 by evaluating the integral in A_1 and B_1 using the sum of the integrand for every hour from t_0 to $t_0 + T$. Similarly, if $T = 48(h)$ or $T = 96(h)$, then the theoretical value is 900 Wh which is obtained from Proposition 8. Due to the step size 10 Wh used in the choice of $E_{\max}$, these theoretical values are quite consistent with results obtained via simulations.

5.2. Constant Loads

We now study how the cost function J of the optimization problem in Eq. (5) changes as a function of $E_{\max}$ with a constant load, and the load is used from t_0 to $t_0 + T$ to satisfy Assumption 2. We fix the load to be $P_{\text{load}} = 200$ W, and increase the battery capacity $E_{\max}$ from 0 to 1500 Wh with the step size 10 Wh. We solve the optimization problem in Eq. (5) via linear programming using the CPlex sover [21]. If $T = 24(h)$ (or $T = 48(h)$, $T = 96(h)$, respectively), the plot of the minimum costs versus $E_{\max}$ is given in Fig. 9(a) (or (b), (c), respectively). The plots confirm the result in Proposition 1, i.e., the minimum cost is a decreasing function of $E_{\max}$, and also show the existence of the unique $E^{c}_{\max}$.

Now we validate the results in Propositions 6, 7, 8, and 9. We vary the load from 0 to 1500 W with the step size 100 W, and for each P_{load} we calculate the $E^{c}_{\max}$ and the minimum cost corresponding to the E^{c}_{max}. In Fig. 10(a), the left figure shows how E^{c}_{max} changes as a function of the load for $T = 24(h)$, and the right figure shows the corresponding minimum costs. The plot in the left figure is consistent with the result in Proposition 7 except that the maximum of E^{c}_{max} is not unique. This is due to the fact that the load is chosen to be discrete with step size 100 W. The right figure is consistent with the intuition that when the load is increasing, more electricity needs to be purchased from the grid (resulting in a higher cost). Note that the blue solid curve corresponds to the costs with E^{c}_{max}, while the red dotted curve corresponds to J_{max}, i.e., the costs without battery. The plots for E^{c}_{max} and the minimum cost for $T = 48(h)$ and $T = 96(h)$ are shown in Fig. 10(b) and (c). The plots in the left figures of Fig. 10(b) and (c) are consistent with the result in Proposition 9. Note that as T increases, the critical load P^{c}_{load} decreases as shown in the left figures of Fig. 10. One observation on the left figures of Fig. 10 is that E^{c}_{max} increases roughly linearly with respect to the load when the load is small. The justification is that when the load is small, E^{c}_{max} is determined by B_1 in Fig. 3 (or B_2 in Fig. 5 for multiple days) and B_1 (or B_2) increases roughly linearly with respect to the load as can be seen from Fig. 3 (or Fig. 5).

Now we examine the results in Proposition 6. For $T = 24(h)$, we evaluate the integral in A_1 and B_1 using the sum of the integrand for every hour from t_0 to $t_0 + T$ given a fixed load, and then obtain E^{c}_{max}; this value is denoted as the theoretical value. The theoretical value is plotted as the red curve (with the circle marker) in the left plot of Fig. 11(a). The value E^{c}_{max} calculated based on simulations is plotted as the blue curve (with the square marker) in the left plot of Fig. 11(a). In the right plot of Fig. 11(b), we plot the difference between the value obtained via simulations and the theoretical value. Note that the value obtained via simulations is always larger than or equal to the theoretical value because E_{max} is chosen to be discrete with step size 10 Wh. The differences are always smaller than[4] 9 Wh, which confirms that the theoretical value is very consistent with the value obtained via simulations. The same conclusion holds for $T = 48(h)$, as shown in Fig. 11(b). For $T = 96(h)$, the largest difference is around 70 Wh as shown in Fig. 11(c); this is more likely due to the slight variation in the PV generation for different days. Note that the differences for $\frac{10}{16} = \frac{5}{8}$ of the load values (which range from 0 to 1500 W with the step size 100 W) are within 10 Wh.

6. Conclusions

In this paper, we studied the problem of determining the size of battery storage for grid-connected PV systems. We proposed an upper bound on the storage size, and showed that the upper bound is achievable for certain scenarios. For the case with ideal PV generation and constant load, we characterized the exact storage size, and also showed how the storage size changes as the constant load changes; these results are consistent with the results obtained via simulations.

There are several directions for future research. First, the dynamic time-of-use pricing of the electricity purchase from the grid could be taken into account. Large businesses usually pay time-of-use electricity rates, but with increased deployment of smart meters and electric vehicles some utility companies are moving towards different prices for residential electricity purchase at different times of the day (for example, SDG&E has the peak, semipeak, offpeak prices for a day in the summer season [20]). New results (and probably new techniques) are necessary to

[4] The step size for E_{max} is 10 Wh, so the difference between the value obtained via simulations and the theoretical value is expected to be within 10 assuming the theoretical value is correct.

deal with dynamic pricing. Second, we would like to study how batteries with a fixed capacity can be utilized (e.g., via serial or parallel connections) to implement the critical battery capacity for practical applications. Last, we would also like to extend the results to wind energy storage systems, and consider battery parameters such as round-trip charging efficiency, degradation, and costs.

7. Acknowledgements

This work was funded by NSF grant ECCS-1232271.

Appendix A. Proof to Proposition 1

Given $E_{\max}^1$, suppose a feasible control $u^1(t)$ achieves the minimum electricity purchase cost $J(E_{\max}^1)$ and the corresponding state x is $x^1(t)$. Since $|x^1(t)| \leq E_{\max}^1 < E_{\max}^2$, $u^1(t)$ is also a feasible control for problem (5) with the state constraint $E_{\max}^2$ and satisfying Rule 1, and results in the cost $J(E_{\max}^1)$. Since $J(E_{\max}^2)$ is the minimal cost over the set of all feasible controls which include $u^1(t)$, we must have $J(E_{\max}^1) \geq J(E_{\max}^2)$.

Appendix B. Proof to Proposition 3

Condition (i) holds. Since $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \leq 0$, we have $P_{\text{load}}(t) - P_{\text{pv}}(t) \geq 0$. Denote the integrand in J of Eq. (5) as α, i.e., $\alpha(t) = C_{\text{gp}} \max(0, P_{\text{load}}(t) - P_{\text{pv}}(t) + u(t))$. If $P_{\text{load}}(t) - P_{\text{pv}}(t) = 0$, then we could choose $u(t) = 0$ to make α to be 0. If $P_{\text{load}}(t) - P_{\text{pv}}(t) > 0$, we could choose $P_{\text{pv}}(t) - P_{\text{load}}(t) \leq u(t) < 0$ to decrease α, i.e., by discharging the battery. However, since $x(t_0) = -E_{\max}$, there is no electricity stored in the battery at the initial time. To be able to discharge the battery, it must have been charged previously. Following Rule 1, the electricity stored in the battery should only come from surplus PV generation. However, there is no surplus PV generation at any time because $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \leq 0$. Therefore, the cost is not reduced by choosing $P_{\text{pv}}(t) - P_{\text{load}}(t) \leq u(t) < 0$. In other words, u can be chosen to be 0. In both cases, $u(t)$ can be 0 for any $t \in [t_0, t_0 + T]$ without increasing the cost, and thus, no battery is necessary. Therefore, $E_{\max}^{\text{c}} = 0$.

Condition (ii) holds. Since $\forall t \in [t_0, t_0 + T]$, $P_{\text{pv}}(t) - P_{\text{load}}(t) \geq 0$, we have $P_{\text{load}}(t) - P_{\text{pv}}(t) \leq 0$. Denote the integrand in J as α, i.e., $\alpha(t) = C_{\text{gp}} \max(0, P_{\text{load}}(t) - P_{\text{pv}}(t) + u(t))$. If $P_{\text{load}}(t) - P_{\text{pv}}(t) \leq 0$, we could choose $u(t) = 0$, and then $\alpha = \max(0, P_{\text{load}}(t) - P_{\text{pv}}(t) + u(t)) = \max(0, P_{\text{load}}(t) - P_{\text{pv}}(t)) = 0$. Since $u(t)$ can be 0 for any $t \in [t_0, t_0 + T]$ without increasing the cost, no battery is necessary. Therefore, $E_{\max}^{\text{c}} = 0$.

Condition (iii) holds. S_1 is the set of time instants at which there is extra amount of electric power that is generated from PV after supplying the load, while S_2 is the set of time instants at which the PV generated power is insufficient to supply the load. According to Rule 1, at time t, the battery could get charged only if $t \in S_1$, and could get discharged only if $t \in S_2$. If $\forall t_1 \in S_1$, $\forall t_2 \in S_2$, $t_2 < t_1$ implies that even if the extra amount of electricity generated from PV is stored in a battery, there is no way to use the stored electricity to supply the load. This is because the electricity is stored after the time instants at which battery discharging can be used to strictly decrease the cost and initially there is no electricity stored in the battery. Therefore, the costs are the same for the scenario with battery and the scenario without battery, and $E_{\max}^{\text{c}} = 0$.

Appendix C. Proof to Proposition 4

It can be shown, via contradiction, that under Assumption 1, $A > 0$ and $B > 0$, which imply that $\frac{\min(A,B)}{2} > 0$.

We show $E^{\mathrm{c}}_{\max} > 0$ via contradiction. Since $E^{\mathrm{c}}_{\max} \geq 0$, we need exclude the case $E^{\mathrm{c}}_{\max} = 0$. Suppose $E^{\mathrm{c}}_{\max} = 0$. If we choose $E_{\max} > E^{\mathrm{c}}_{\max} = 0$, $J(E_{\max}) < J(E^{\mathrm{c}}_{\max})$ because under Assumption 1 a battery can store the extra PV generated electricity first and then use it later on to strictly reduce the cost compared with the case without a battery (i.e., the case with $E_{\max} = 0$). A contradiction to the definition of $E^{\mathrm{c}}_{\max}$.

To show $E^{\mathrm{c}}_{\max} \leq \frac{\min(A,B)}{2}$, it is sufficient to show that if $E_{\max} \geq \frac{\min(A,B)}{2}$, then $J(E_{\max}) = J(\frac{\min(A,B)}{2})$. There are two cases depending on if $A \leq B$ or not:

- $A \leq B$. Then $\min(A, B) = A$. At time t, $\max(0, P_{\mathrm{pv}}(t) - P_{\mathrm{load}}(t))$ is the extra amount of electric power that is generated from PV after supplying the load, and

$$\min(P_{\mathrm{Bmax}}, \max(0, P_{\mathrm{pv}}(t) - P_{\mathrm{load}}(t)))$$

  is the extra amount of electric power that is generated from PV after supplying the load and can be stored in a battery subject to the maximum charging rate. Then

$$A = \int_{t_0}^{t_0+T} \min(P_{\mathrm{Bmax}}, \max(0, P_{\mathrm{pv}}(t) - P_{\mathrm{load}}(t)))dt$$

  is the maximum total amount of extra electricity that can be stored in a battery while taking the battery charging rate into account. Even if $2E_{\max} \geq A$, i.e., $E_{\max} \geq \frac{A}{2}$, the amount of electricity that can be stored in the battery cannot exceed A. Therefore, any control that is feasible with $|x(t)| \leq E_{\max}$ is also feasible with $|x(t)| \leq \frac{A}{2}$. Therefore, $J(E_{\max}) = J(\frac{A}{2}) = J(\frac{\min(A,B)}{2})$;

- $A > B$. Then $\min(A, B) = B$. At time t, $\max(0, P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t))$ is the amount of electric power that is necessary to satisfy the load (and could be supplied by either battery power or grid purchase), and

$$\min(-P_{\mathrm{Bmin}}, \max(0, P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t)))$$

  is the amount of electric power that can potentially be discharged from a battery to supply the load subject to the maximum discharging rate (in other words, if $P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t) > -P_{\mathrm{Bmin}}$, electricity must be purchased from the grid). Then

$$B = \int_{t_0}^{t_0+T} \min(-P_{\mathrm{Bmin}}, \max(0, P_{\mathrm{load}}(t) - P_{\mathrm{pv}}(t)))dt$$

  is the maximum total amount of electricity that is necessary to be discharged from the battery to satisfy the load while taking the battery discharging rate into account. When $2E_{\max} \geq B$, i.e., $E_{\max} \geq \frac{B}{2}$, the amount of electricity that can be charged can exceed B because $A > B$; however, the amount of electricity that is strictly necessary to be (and, at the same time, can be) discharged does not exceed B. In other words, if the stored electricity in the battery exceeds this amount B, the extra electricity cannot help reduce the cost because it either cannot be discharged or is not necessary. Therefore, any control that minimizes the total cost with the battery capacity being B also minimizes the total cost with the battery capacity being $2E_{\max}$. Therefore, $J(E_{\max}) = J(\frac{B}{2}) = J(\frac{\min(A,B)}{2})$.

Appendix D. Proof to Proposition 5

From Proposition 4, we have $E^{c}_{\max} \leq \frac{\min(A,B)}{2}$. To prove $E^{c}_{\max} = \frac{\min(A,B)}{2}$, we show that $E^{c}_{\max} < \frac{\min(A,B)}{2}$ is impossible via contradiction. Suppose $E^{c}_{\max} < \frac{\min(A,B)}{2}$. If $\forall t_1 \in S_1$, $\forall t_2 \in S_2$, $t_1 < t_2$, then during the time interval $[t_0, t_0 + T]$, the battery is first charged, and then discharged following Rule 1. In other words, there is no charging after discharging. There are two cases depending on A and B:

- $A \leq B$. In this case, $E^{c}_{\max} < \frac{A}{2}$, i.e., $2E^{c}_{\max} < A$. If the battery capacity is $2E^{c}_{\max}$, then the amount of electricity $A - 2E^{c}_{\max} > 0$ (which is generated from PV) cannot be stored in the battery. If we choose the battery capacity to be A, this extra amount can be stored and used later on to strictly decrease the cost because $A \leq B$. Therefore, $J(\frac{A}{2}) < J(E^{c}_{\max})$. A contradiction to the definition of $E^{c}_{\max}$. In this case, if $E_{\max}$ is chosen to be $\frac{A}{2}$, then the battery is first charged with A amount of electricity, and then completely discharged before (or at) $t_0 + T$ because $A \leq B$. Therefore, we have $x(t_0 + T) = -E_{\max}$.
- $A > B$. In this case, $E^{c}_{\max} < \frac{B}{2}$, i.e., $2E^{c}_{\max} < B$. If the battery capacity is $2E^{c}_{\max}$, at most $2E^{c}_{\max} < B < A$ amount of PV generated electricity can be stored in the battery. Therefore, the amount of electricity $B - 2E^{c}_{\max} > 0$ must be purchased from the grid to supply the load. If we choose the battery capacity to be B, the amount of electricity $B - 2E^{c}_{\max}$ purchased from the grid can be provided by the battery because the battery can be charged with the amount of electricity B (since $A > B$), and thus the cost can be strictly decreased. Therefore, $J(\frac{B}{2}) < J(E^{c}_{\max})$. A contradiction to the definition of $E^{c}_{\max}$. In this case, if $E_{\max}$ is chosen to be $\frac{B}{2}$, then the battery is first charged with B amount of electricity (that is to say, not all extra electricity generated from PV is stored in the battery since $A > B$), and then completely discharged at time $t_0 + T$. Therefore, we also have $x(t_0 + T) = -E_{\max}$.

Appendix E. Proof to Proposition 6

Due to Assumption 2, $P_{\text{load}}(t)$ intersects with $P_{\text{pv}}(t)$ at two time instants for $T = 24(h)$; the smaller time instant is denoted as t_1, and the larger is denoted as t_2, as shown in Fig. 3. It can be verified that $P_{\text{pv}}(t) > P_{\text{load}}$ for $t \in (t_1, t_2)$ and $P_{\text{pv}}(t) < P_{\text{load}}$ for $t \in [0, t_1) \cup (t_2, 24]$ following Assumption 2. For $t \in [0, t_1)$, a battery could only get discharged following Rule 1; however, it cannot be discharged because $x(0) = -E_{\max}$. Therefore, $u(t)$ can be 0 while achieving the lowest cost for the time period $[0, t_1)$. Then the objective function of the optimization problem in Eq. (5) can be rewritten as

$$
\begin{aligned}
J &= \min \int_0^{24} C_{\text{gp}} \max(0, P_{\text{load}} - P_{\text{pv}}(\tau) + u(\tau)) d\tau \\
&= J_0 + J_1 \,,
\end{aligned}
$$

where $J_0 = \int_0^{t_1} C_{\text{gp}}(P_{\text{load}} - P_{\text{pv}}(\tau)) d\tau$ is a constant which is independent of the control u, and

$$
J_1 = \min \int_{t_1}^{24} C_{\text{gp}} \max(0, P_{\text{load}} - P_{\text{pv}}(\tau) + u(\tau)) d\tau \,.
$$

In other words, the optimization problem is essentially the same as minimizing J_1 for $t \in [t_1, 24]$; accordingly, the critical value $E^{c}_{\max}$ will be the same since the battery is not used for the time interval $[0, t_1]$. For the optimization problem with the cost function J_1 under Assumption 2, $S_1 = (t_1, t_2)$ and $S_2 = (t_2, 24)$ according to Eqs. (7) and (8). Since $\forall t_1' \in S_1, \forall t_2' \in S_2, t_1' < t_2 < t_2'$, the conditions in Proposition 5 are satisfied. Thus, we have $E^{c}_{\max} = \frac{\min(A,B)}{2}$, where

$$\begin{aligned} A &= \int_{t_1}^{24} \min(P_{\text{Bmax}}, \max(0, P_{\text{pv}}(t) - P_{\text{load}}))dt \\ &= \int_{t_1}^{t_2} \min(P_{\text{Bmax}}, \max(0, P_{\text{pv}}(t) - P_{\text{load}}))dt, \end{aligned}$$

which is essentially A_1, and

$$\begin{aligned} B &= \int_{t_1}^{24} \min(-P_{\text{Bmin}}, \max(0, P_{\text{load}} - P_{\text{pv}}(t)))dt \\ &= \int_{t_2}^{24} \min(-P_{\text{Bmin}}, \max(0, P_{\text{load}} - P_{\text{pv}}(t)))dt \ , \end{aligned}$$

which is essentially B_1. Thus the result holds.

Appendix F. Proof to Proposition 7

Let $f := A_1 - B_1$, where A_1 and B_1 are defined in Eqs. (11) and (12). Note that f is a function of P_{load}. If $P_{\text{load}} = 0$, then

$$A_1 = \int_0^{24} \min(P_{\text{Bmax}}, \max(0, P_{\text{pv}}(t)))dt > 0 \ ,$$

according to Eq. (11), and

$$B_1 = \int_{t_{\max}}^{24} \min(-P_{\text{Bmin}}, \max(0, -P_{\text{pv}}(t)))dt = 0 \ ,$$

according to Eq. (12). Therefore, $f(0) = A_1(0) - B_1(0) > 0$. If $P_{\text{load}} = P^{\max}_{\text{pv}}$, then

$$A_1 = \int_0^{24} \min(P_{\text{Bmax}}, \max(0, P_{\text{pv}}(t) - P^{\max}_{\text{pv}}))dt = 0,$$

and

$$B_1 = \int_{t_{\max}}^{24} \min(-P_{\text{Bmin}}, \max(0, P^{\max}_{\text{pv}} - P_{\text{pv}}(t)))dt > 0 \ .$$

Therefore, $f(P^{\max}_{\text{pv}}) = A_1(P^{\max}_{\text{pv}}) - B_1(P^{\max}_{\text{pv}}) < 0$. In addition, since f is an integral of a continuous function of P_{load}, f is differentiable with respect to P_{load}, and the derivative is given as

$$\frac{df}{dP_{\text{load}}} = \frac{dA_1}{dP_{\text{load}}} - \frac{dB_1}{dP_{\text{load}}} \ .$$

Since for $P_{\text{load}} \in (0, P^{\max}_{\text{pv}})$,

$$\begin{aligned} \frac{dA_1}{dP_{\text{load}}} &= \int_0^{24} (-1) \times I\{0 < P_{\text{pv}}(t) - P_{\text{load}} \le P_{\text{Bmax}}\}dt \\ &< 0 \ , \end{aligned}$$

and

$$\frac{dB_1}{dP_{\text{load}}} = \int_{t_{\max}}^{24} 1 \times I\{0 < P_{\text{load}} - P_{\text{pv}}(t) \leq -P_{\text{Bmin}}\}dt$$
$$> 0 \, ,$$

we have $\frac{df}{dP_{\text{load}}} < 0$, where $I\{0 < P_{\text{pv}}(t) - P_{\text{load}} \leq P_{\text{Bmax}}\}$ is the indicator function (i.e., if $0 < P_{\text{pv}}(t) - P_{\text{load}} \leq P_{\text{Bmax}}$, the function has value 1, 0 otherwise). Therefore, f is continuous and strictly decreasing for $P_{\text{load}} \in [0, P_{\text{pv}}^{\max}]$. Since $f(0) > 0$ and $f(P_{\text{pv}}^{\max}) < 0$, there is one and only one value of P_{load} such that f is 0. We denote this value as P_{load}^c and have $A_1(P_{\text{load}}^c) = B_1(P_{\text{load}}^c)$.

If $P_{\text{load}} \in [0, P_{\text{load}}^c)$, $f > 0$, i.e., $A_1 > B_1$. Therefore, $E_{\max}^{\text{c}} = \frac{B_1}{2}$. Since $\frac{dB_1}{dP_{\text{load}}} > 0$, $E_{\max}^{\text{c}}$ increases continuously (since B_1 is differentiable with respect to P_{load}) and monotonically from 0 to the value $\frac{B_1(P_{\text{load}}^c)}{2}$. On the other hand, if $P_{\text{load}} \in (P_{\text{load}}^c, P_{\text{pv}}^{\max}]$, $f < 0$, i.e., $A_1 < B_1$. Therefore, $E_{\max}^{\text{c}} = \frac{A_1}{2}$. Since $\frac{dA_1}{dP_{\text{load}}} < 0$, $E_{\max}^{\text{c}}$ decreases continuously (since A_1 is differentiable with respect to P_{load}) and monotonically from the value $\frac{A_1(P_{\text{load}}^c)}{2} = \frac{B_1(P_{\text{load}}^c)}{2}$ to 0. Therefore, $E_{\max}^{\text{c}}$ achieves its maximum at P_{load}^c. This completes the proof.

Appendix G. Proof to Proposition 8

Due to Assumption 2, $P_{\text{load}}(t)$ intersects with $P_{\text{pv}}(t)$ at $2k$ time instants for $T = k \times 24(h)$; we denote the set of these time instants as $T_{crossing} := \{t \in [0, k \times 24] \mid P_{\text{pv}}(t) = P_{\text{load}}\}$. We sort the time instants in an ascending order and denote them as $t_1, t_2, t_3, ..., t_{2i-1}, t_{2i}, ..., t_{2k-1}, t_{2k}$, where $2 \leq i \leq k$. Following Rule 1, at time t, a battery could get charged only if $t \in (t_1, t_2) \cup (t_3, t_4) \cup \cdots (t_{2k-1}, t_{2k})$, and could get discharged only if $t \in (0, t_1) \cup (t_2, t_3) \cup (t_4, t_5) \cup \cdots (t_{2k}, k \times 24)$. As shown in the proof to Proposition 6, $u(t)$ can be zero for $t \in (0, t_1)$, and results in the lowest cost $J_0 = \int_0^{t_1} C_{\text{gp}}(P_{\text{load}} - P_{\text{pv}}(\tau))d\tau$, which is a constant. At time t_1, there is no charge in the battery. Then the battery is operated repeatedly by charging first if $t \in (t_{2i-1}, t_{2i})$ and then discharging if $t \in (t_{2i}, t_{2i+1})$ for $i = 1, 2, ..., k$ and $t_{2k+1} = k \times 24$. Naturally, we could group the charging interval (t_{2i-1}, t_{2i}) with the discharging interval $t \in (t_{2i}, t_{2i+1})$ to form a complete battery operating cycle in the interval (t_{2i-1}, t_{2i+1}).

Now the objective function of the optimization problem in Eq. (5) satisfies

$$J = \min_u \int_0^{k\times 24} C_{\text{gp}} \max(0, P_{\text{load}} - P_{\text{pv}}(\tau) + u(\tau))d\tau$$
$$= J_0 + \min_u (\sum_{i=1}^{k-1} L_i + L_k) \, ,$$

where

$$L_i = \int_{t_{2i-1}}^{t_{2i+1}} C_{\text{gp}} \max(0, P_{\text{load}} - P_{\text{pv}}(\tau) + u(\tau))d\tau \, ,$$

for $i = 1, 2, ..., k - 1$, and

$$L_k = \int_{t_{2k-1}}^{t_{2k+1}} C_{\text{gp}} \max(0, P_{\text{load}} - P_{\text{pv}}(\tau) + u(\tau))d\tau \, .$$

Note that given a certain $E_{\max}$, if the battery charge at the end of the first battery operating cycle is larger than 0 (i.e., $x(t_3) > -E_{\max}$), then $E_{\max} > E^{\mathrm{c}}_{\max}$. This can be argued as follows. If the battery charge at the end of the first cycle is larger than 0 (this also implies that the battery charge at the end of the ith cycle is also larger than 0 due to periodic PV generations and loads), i.e., there is more PV generation than demand in the time interval (t_1, t_3), then $E_{\max}$ can be strictly reduced to a smaller capacity so that $x(t_3) = -E_{\max}$ without increasing the electricity purchase cost in the interval (t_1, t_3). Due to periodicity of the PV generation and the load, the smaller $E_{\max}$ can be used for the interval (t_{2i-1}, t_{2i+1}) for $i = 2, ..., k-1$ without increasing the electricity purchase cost. Therefore, this $E_{\max}$ must be larger than $E^{\mathrm{c}}_{\max}$. In other words, if $E^{\mathrm{c}}_{\max}$ is used, then $x(t_{2i+1})$ for $i = 1, 2, ..., k-1$ has to be $-E^{\mathrm{c}}_{\max}$, i.e., no charge left at the end of each operating cycle. Now we only consider $E_{\max}$ such that at the end of each operating cycle $x(t_{2i+1}) = -E_{\max}$ for $i = 1, 2, ..., k-1$ (necessarily, $E^{\mathrm{c}}_{\max}$ is smaller than or equal to any such $E_{\max}$). For such $E_{\max}$, the control actions for each operating cycle are completely decoupled[5]. Therefore, the total cost J can be rewritten as

$$J = J_0 + \sum_{i=1}^{k-1} J_i + J_k \ ,$$

where $J_i = \min_u L_i$ for $i = 1, 2, ..., k$.

Now we focus on J_1. For the optimization problem with the cost function J_1 under Assumption 2, $S_1 = (t_1, t_2)$ and $S_2 = (t_2, t_3)$ according to Eqs. (7) and (8). Since $\forall t_1' \in S_1$, $\forall t_2' \in S_2$, $t_1' < t_2 < t_2'$, the conditions in Proposition 5 are satisfied. Thus, we have $E^{\mathrm{c}}_{\max}(1) = \frac{\min(A,B)}{2}$, where $E^{\mathrm{c}}_{\max}(1)$ is the $E^{\mathrm{c}}_{\max}$ when we only consider the cost function J_1,

$$\begin{aligned} A &= \int_{t_1}^{t_3} \min(P_{\mathrm{Bmax}}, \max(0, P_{\mathrm{pv}}(t) - P_{\mathrm{load}}))dt \\ &= \int_{t_1}^{t_2} \min(P_{\mathrm{Bmax}}, \max(0, P_{\mathrm{pv}}(t) - P_{\mathrm{load}}))dt \ , \end{aligned}$$

which is essentially A_2, and

$$\begin{aligned} B &= \int_{t_1}^{t_3} \min(-P_{\mathrm{Bmin}}, \max(0, P_{\mathrm{load}} - P_{\mathrm{pv}}(t)))dt \\ &= \int_{t_2}^{t_3} \min(-P_{\mathrm{Bmin}}, \max(0, P_{\mathrm{load}} - P_{\mathrm{pv}}(t)))dt \ , \end{aligned}$$

which is essentially B_2. Thus we have $E^{\mathrm{c}}_{\max}(1) = \frac{\min(A_2,B_2)}{2}$. Based on Proposition 5, we also know that $x(t_3) = -E^{\mathrm{c}}_{\max}(1)$. Thus, this $E^{\mathrm{c}}_{\max}(1)$ satisfies the requirement that at the end of the operating cycle there is no charge left.

For the cost function J_2, the optimization problem is essentially the same as the problem with the cost function J_1 because

[5]Note that the control action for $t \in (t_{2i-1}, t_{2i})$ and the control action for $t \in (t_{2i}, t_{2i+1})$ for $i = 1, 2, ..., k$ are coupled in the sense that battery can not be discharged if at time t_{2i} there is no charge in the battery. In general, the control action for $t \in (t_{2i}, t_{2i+1})$ and the control action for $t \in (t_{2i+1}, t_{2i+2})$ for $i = 1, k-1$ can also be coupled if at time t_{2i+1} there is extra charge left in the battery because the extra charge will affect the charging action in the interval $t \in (t_{2i+1}, t_{2i+2})$. Here, there is no such coupling for the latter case when we restrict $E_{\max}$ so that at the end of each operating cycle there is no charge left.

- $P_{\rm pv}(t) = P_{\rm pv}(t-24)$ for $t \in [t_3, t_5]$ because $P_{\rm pv}(t)$ is periodic with period 24 hours. Note that $t - 24 \in [t_1, t_3]$;
- $P_{\rm load}(t)$ is a constant; and
- there is no charge left at t_3.

In other words, there is no difference between the optimization problem with the cost function J_2 and the one with J_1 other than the shifting of time t by 24 hours. Therefore, the $E^{\rm c}_{\rm max}(2)$ will be the same as $E^{\rm c}_{\rm max}(1)$. The same reasoning applies to the optimization problem with the cost function J_i for $i = 3, ..., k-1$. Therefore, we have $E^{\rm c}_{\rm max}(i) = E^{\rm c}_{\rm max}(1)$ for $i = 2, ..., k-1$.

For the optimization problem J_k, there is no charge left at time $2k-1$. This problem is essentially the same as the problem studied in the proof to Proposition 6 with the cost function J_1 except the shifting of time t by $(k-1) \times 24$ hours. The solution $E^{\rm c}_{\rm max}(k)$ is given as $\frac{\min(A_k,B_k)}{2}$, where

$$A_k = \int_{t_{2k-1}}^{t_{2k}} \min(P_{\rm Bmax}, \max(0, P_{\rm pv}(t) - P_{\rm load}))dt \ ,$$

and

$$B_k = \int_{t_{2k}}^{t_{2k+1}} \min(-P_{\rm Bmin}, \max(0, P_{\rm load} - P_{\rm pv}(t)))dt \ .$$

Note that $A_k = A_2$, and $B_k < B_2$. If we choose $E_{\rm max} = \frac{\min(A_2,B_2)}{2}$ which is larger than or equal to $E^{\rm c}_{\rm max}(k)$, we have $J_k(E_{\rm max}) = J_k(E^{\rm c}_{\rm max}(k))$.

Now we claim that $E^{\rm c}_{\rm max}$ when considering the cost function J is exactly $\frac{\min(A_2,B_2)}{2}$. If we choose $E_{\rm max} < \frac{\min(A_2,B_2)}{2}$, then $J(E_{\rm max}) > J(\frac{\min(A_2,B_2)}{2})$ by an argument similar to the one in Proposition 5. On the other hand, if we choose $E_{\rm max} \geq \frac{\min(A_2,B_2)}{2}$, then $J(E_{\rm max}) = J(\frac{\min(A_2,B_2)}{2})$. Therefore, $E^{\rm c}_{\rm max}$ to the optimization problem with the cost function J is $\frac{\min(A_2,B_2)}{2}$.

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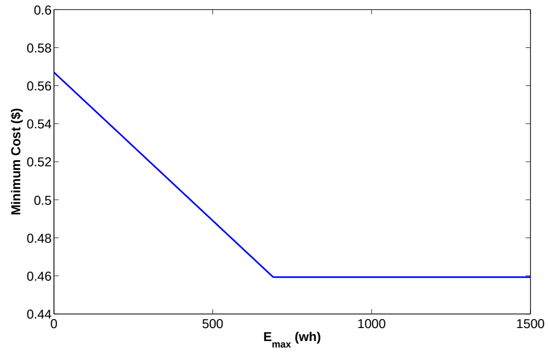
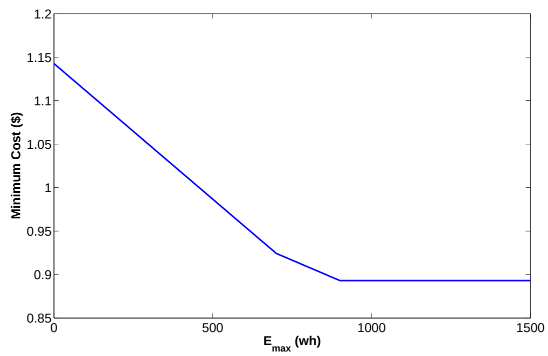


(a) One day.



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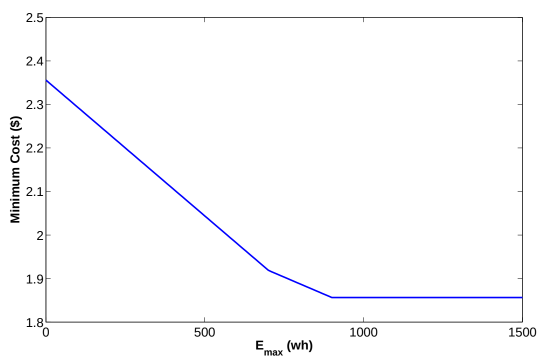
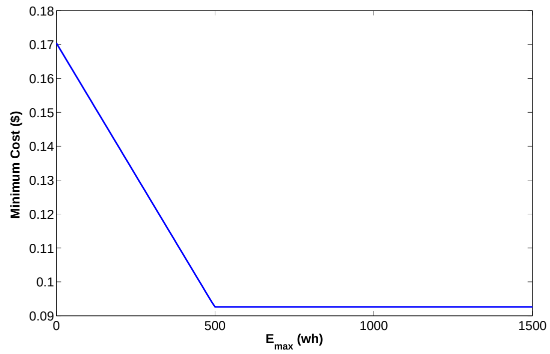


(c) Four days.

Figure 8: Plots of minimum costs versus $E_{\max}$ obtained via simulations given the load profile in Fig. 7 and $P_{\mathrm{Bmax}} = 200\ W$.



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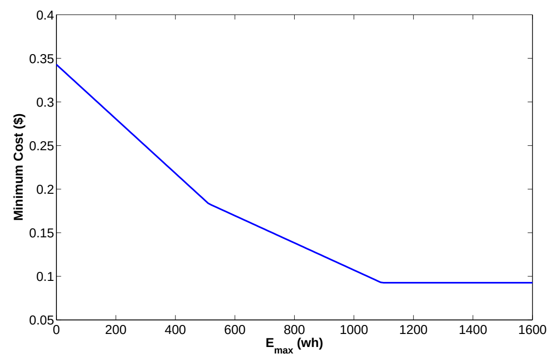

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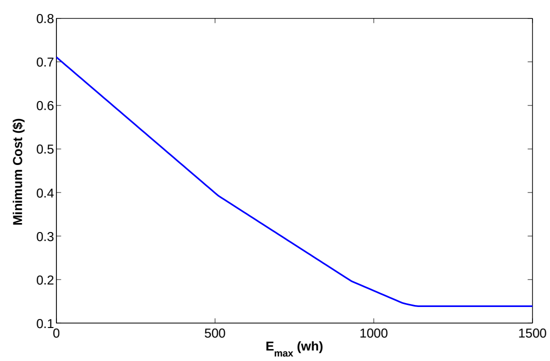
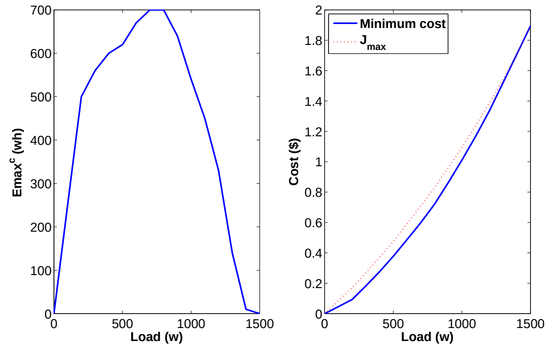


(c) Four days.

Figure 9: Plots of minimum costs versus $E_{\max}$ obtained via simulations given $P_{\text{load}} = 200\ W$ and $P_{\text{Bmax}} = 200\ W$.



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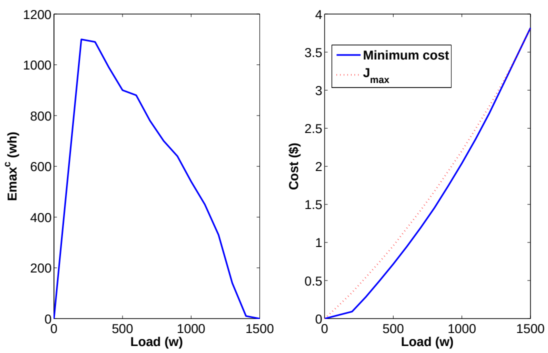

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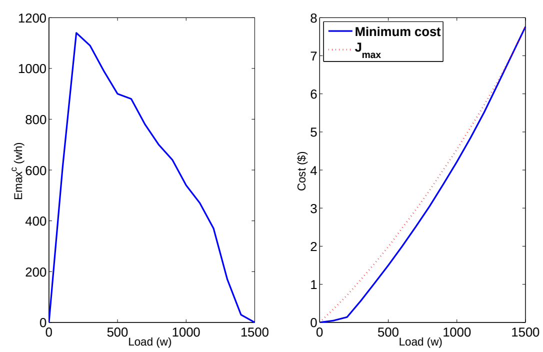
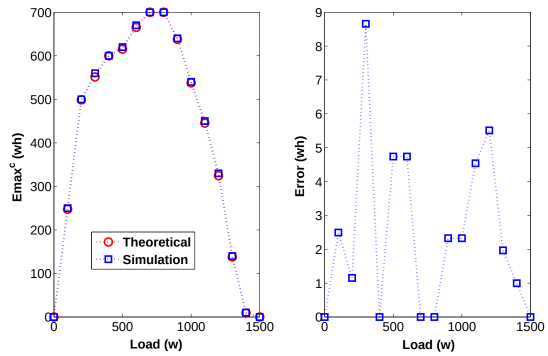


(c) Four days.

Figure 10: $E^{c}_{\max}$ (left), and costs (both $J_{\max}$ and the cost corresponding to $E^{c}_{\max}$, right) versus the fixed load obtained via simulations for $P_{Bmax} = 200\ W$.



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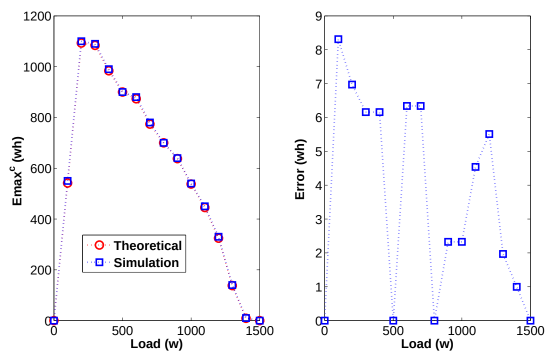

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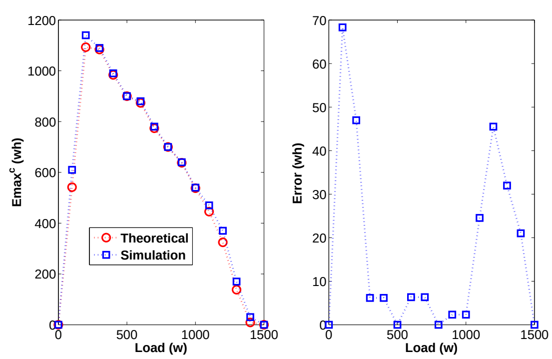

(c) Four days.

Figure 11: Plots of $E^{c}_{\max}$ versus the fixed load for the theoretical value (the red curve with the circle marker) and the value obtained via simulations (the blue curve with the square marker).