Step Size Analysis in Discrete-time Dynamic Average Consensus

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Abstract— **This paper deals with the problem of reaching the average consensus of a set of time-varying reference signals in a distributed manner. We analyze the approach initially presented in [1], giving an alternative proof of convergence which leads to larger, more realistic bounds on the step sizes that guarantee a steady-state error upper-bounded by a given constant. The interest of the new results appear when the algorithm is used in real networks, where there are constraints in the communication rate between the nodes. We derive the bounds for the cases of fixed and time-varying communication topologies, as well as for different orders of the consensus algorithm. We demonstrate that our bounds always allow substantially bigger step sizes independently of the number of nodes or the network topology. Moreover, for a fixed step size and steady-state error, we show how there is a corresponding algorithm that can guarantee that the error is no larger than the desired one, using that step size. Finally, simulation results corroborates the theoretical findings of the paper.**

I. Introduction

This paper studies the problem of reaching the average of a set of time-varying reference signals in a distributed manner, the so called distributed dynamic consensus problem. Solutions to this problem find numerous applications in diverse fields such as sensor fusion [2], cooperative control [3], decision making with changing opinions [4] and Kalman filtering [5].

In the dynamic consensus problem, each node of the network can measure a different time-varying signal and the objective of the network is to track the average of all the signals measured by the nodes. Most of the solutions in the literature consider continuous time algorithms [2], [3], [6]–[8]. Frequency domain analysis is used to guarantee zero steady-state error of ramp inputs in [2]. A PI-dynamic consensus algorithm is presented in [6] and posteriorly extended in [7]. The approach presented in [3] considers a common reference input for all the nodes in the network. Recently, the authors of [8] have introduced two continuous-time algorithms to solve the dynamic consensus problem. The results are proven to work for differentiable signals, and for fixed topologies, while showing good adaptation to discontinous changes in simulation. Although all these approaches can be discretized using, e.g., Euler method, the step size they can afford is usually limited.

On the other hand, discrete-time approaches are more appealing because they usually can handle larger step-size measurements of the reference signals and, thus, have lower communication requirements. To the best of the authors knowledge, the only pure discrete-time approaches are the ones in [1], [9] (i.e., they do not arise from a discretization). The convergence analysis of [1] relies on input-to-output stability, providing bounds on the step size the nodes can choose to guarantee a desired steady-state error with respect to the average. The approach in [9] is able to reach dynamic consensus in minimal time, provided that the conditions on the step size given in [1] are satisfied. Unfortunately, the bounds given in the former paper are overly conservative, which means that, for small steady-state errors, the required step sizes are almost zero. Since communications between the nodes usually demand time, for a real implementation of the algorithm clearly it is a good advantage to be able to choose large values of the step size.

Our contribution in this paper is a more precise analysis of convergence of the aforementioned class of algorithms. This allows us to obtain very much improved bounds on the step sizes needed to guarantee small steady-state errors, several orders of magnitude larger than the original ones. This is of special interest in real networks, where the nodes cannot communicate very often.

We express the algorithm as a sum of static consensus processes, analyzing the convergence of each of them by means of the eigenvalues of the weight matrices. We demonstrate that the first-order algorithm is able to track the average of the inputs with an arbitrarily small error using a larger step size than the one initially theorized. Compared to [1], we also demonstrate that by increasing the order of the consensus algorithm, larger and larger step sizes are permitted yet guaranteing a desired steady-state error. In other words, the step size can be arbitrarily large while the steady-state error is arbitrarily small, by choosing a sufficiently large order consensus algorithm. Finally, we demonstrate the performance of our results with simulations.

The rest of the paper is organized in the following manner: In Section II we describe the dynamic average consensus problem and review the approach presented in [1]. Section III analyzes the first-order consensus algorithm considering a fixed communication topology. The case of time-varying communication topologies is discussed in Section IV, where we also prove that our bounds always allow a larger step size than the existing ones. In Section V we characterize the step size for higher order consensus algorithms. Some simulations

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are shown in Section VI. Finally, the conclusions of this work and future perspectives of research are in Section VII.

II. Preliminaries

We consider a sensor network of N nodes labeled by $i \in \mathcal{V} = \{1, \ldots, N\}$. Communications between the nodes are defined according to a time-varying undirected graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$, where $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ represents the edge set at time instant t. In this way, nodes i and j can communicate at time t if and only if $(i,j) \in \mathcal{E}(t)$. The set of neighbors of node $i \in \mathcal{V}$ is the subset of nodes that can directly communicate with it; i.e., $\mathcal{N}_i(t) = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}(t)\}$.

Each node is able to measure a local, continuous physical process, $r_i : \mathbb{R} \to \mathbb{R}$. We denote $\mathbf{r}(t) = (r_1(t), \ldots, r_N(t))^T$ the vector of the signals measured by each node. The final goal of the network is to design a distributed algorithm such that, using only local information, all the nodes are able to eventually track the average of the signal inputs $r_i(t)$, $i \in \{1, \ldots, N\}$. We denote this average by $\bar{r}(t)$,

$$\bar{r}(t) = \frac{1}{N} \sum_i r_i(t).$$

In order to compute $\bar{r}(t)$ each node maintains an estimation $x_i : \mathbb{N} \to \mathbb{R}$, which is updated at discrete times, $n \in \mathbb{N}$. The sample period used by the nodes to estimate $\bar{r}(t)$ is denoted by h. Therefore, the relationship between the continuous time and the discrete time updates is defined by $nh = t$. The objective is to find the largest value of h that guarantees that $\mathbf{x}(n) \to \bar{r}(nh)\mathbf{1}$ with a sufficiently small error as n evolves, where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^N$.

To achieve this objective we consider the first order linear iteration proposed in [1]. The discrete time update followed by each node is

$$x_i(n+1) = a_{ii}(n)x_i(n) + \sum_{j \neq i} a_{ij}(n)x_j(n) + \Delta r_i(nh), \quad (1)$$

where $a_{ij}(n)$ are the local weights and

$$\Delta r_i(nh) = \Delta r_i(t) = r_i(t) - r_i(t-h).$$

The previous update rule can be put in compact form

$$\mathbf{x}(n+1) = \mathbf{A}(n)\mathbf{x}(n) + \Delta\mathbf{r}(nh), \quad (2)$$

with $\mathbf{x}(n) = (x_1(n), \ldots, x_N(n))^T$, $\mathbf{A}(n) = [a_{ij}(n)]$ the weight matrix, and

$$\Delta\mathbf{r}(t) = (\Delta r_1(t), \ldots, \Delta r_N(t)))^T,$$

the vector with the differences of all the signals.

The local weights must satisfy the following assumption:

***Assumption** 2.1: (Non-degenerate balanced weights)*: There exists a constant $0 < \alpha < 1$ such that, for all $i, j \in \mathcal{V}$ and $n \in \mathbb{N}$,

$$\begin{cases} a_{ij}(n) = 0, \text{ if } j \neq \mathcal{N}_i(n) \\ a_{ij}(n) \in \{0\} \cup [\alpha, 1], \text{ if } j \in \mathcal{N}_i(n) \\ a_{ii}(n) = 1 - \sum_{j \neq i} a_{ij}(n) \geq \alpha, \\ \mathbf{1}^T\mathbf{A}(n) = \mathbf{1}^T, \quad \mathbf{A}(n)\mathbf{1} = \mathbf{1}. \end{cases}$$

This assumption implies that the matrices $\mathbf{A}(n)$ are doubly stochastic. This can be easily satisfied using, e.g., Metropolis Weights [10] or using distributed balancing techniques such as [11], [12].

We also make use of the next assumption on the communication graphs.

***Assumption** 2.2: (Periodical strong connectivity)*: There is some positive integer $B \geq 1$ such that, for all instant $n \geq 0$, the graph $(\mathcal{V}, \mathcal{E}(nh) \cup \mathcal{E}((n+1)h) \cup \ldots \cup \mathcal{E}((n+B-1)h))$ is strongly connected.

Finally, another assumption is imposed over the variation of the signals:

***Assumption** 2.3: (Relatively bounded first-order differences)*: For any $h > 0$ there exist a time invariant constant θ such that

$$\max_i \Delta r_i(nh) - \min_i \Delta r_i(nh) \leq h\theta, \quad \forall n \geq 0.$$

The previous three assumptions are standard in the literature, e.g., [13]–[15] and for further discussion about their meaning we refer the reader to [1].

The following result, also extracted from [1], provides a bound on the step size of iteration (1) to ensure a given steady-state error:

***Theorem** 2.1 (Theorem 3.1 [1]):* Let δ be a positive constant and

$$h_1 = \frac{\delta\alpha^{\frac{1}{2}N(N+1)B+1}}{4\theta(NB-2)}. \quad (3)$$

Under Assumptions 2.1, 2.2 and 2.3 the implementation of (1) with $h \in (0, h_1]$ and initial conditions $x_i(0) = r_i(-h)$, $i \in (1, \ldots, N)$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ.

Unfortunately, the bound in (3) approaches rapidly to zero as the size of the network increases due to the N^2 power of α. In this paper, we derive new bounds that allow us to increase the step size, h, in several orders of magnitude, executing (1) with the same guarantees of achieving a small steady-state error. In order to do this, we provide an alternative analysis to the input-to-output stability of [1], which is based on a decomposition of the algorithm into a sequence of static consensus processes and the convergence properties of each one of them.

III. Step size for fixed communication topologies

In this section, we derive an improved bound for h when the communication topology remains fixed and connected at all times, i.e., $\mathbf{A}(n) = \mathbf{A}$, $\forall n \in \mathbb{N}$. In this scenario, because of Assumption 2.1, $\mathbf{A}$ has one eigenvalue $\lambda_1 = 1$ and the rest eigenvalues satisfy $\lambda_1 > \lambda_2 > \ldots > \lambda_N > -1$. Without loss of generality, we assume that the algebraic connectivity of the network (the second largest eigenvalue in modulus) is defined by the value of $|\lambda_2|$, or in other words $|\lambda_2| > |\lambda_N|$.

***Theorem** 3.1 (Step size for fixed topologies):* Let δ be a positive constant and

$$h^* = \frac{\delta(1-\lambda_2)}{\theta\sqrt{N}} \quad (4)$$

Under Assumptions 2.1 and 2.3 the implementation of (1) with $h \in (0, h^*]$ and initial conditions $x_i(0) = r_i(-h)$, $i \in$

$(1, \ldots, N)$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ.

Proof: Let us denote by $\Delta\bar{r}(nh)$ the variation in the average of the signals, i.e.,

$$\Delta\bar{r}(nh) = \frac{1}{N}\sum_i \Delta r_i(nh). \tag{5}$$

Therefore,

$$\bar{r}(nh) = \Delta\bar{r}(nh) + \Delta\bar{r}((n-1)h) + \ldots + \Delta\bar{r}(0) + \bar{r}(-h). \tag{6}$$

Now, starting with $\mathbf{x}(0) = \mathbf{r}(-h)$, and iterating the expression (2) we obtain

$$\mathbf{x}(n+1) = \mathbf{A}^{n+1}\mathbf{r}(-h) + \sum_{j=0}^{n}\mathbf{A}^{n-j}\Delta\mathbf{r}(jh). \tag{7}$$

Let us define the error with respect to the average at iteration n by $\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty$. Combining equations (6) and (7) yields

$$\begin{aligned}\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1} = \mathbf{A}^{n+1}\mathbf{r}(-h) - \bar{r}(-h)\mathbf{1} + \\ + \sum_{j=0}^{n}[\mathbf{A}^{n-j}\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}].\end{aligned} \tag{8}$$

Since $\mathbf{A}$ is doubly stochastic and compatible with the connected communication graph, then, for all j in the above equation,

$$\lim_{n\to\infty}\mathbf{A}^{n-j}\Delta\mathbf{r}(jh) = \Delta\bar{r}(jh), \tag{9}$$

and the error with respect to the partial averages is upper-bounded by

$$\begin{aligned}&\|\mathbf{A}^{n-j}\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_\infty \leq \\ &\leq \|\mathbf{A}^{n-j}\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_2 \leq \\ &\leq \lambda_2^{n-j}\|\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_2 \leq \\ &\leq \lambda_2^{n-j}\sqrt{N}\|\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_\infty = \\ &= \lambda_2^{n-j}\sqrt{N}\max_i|\Delta r_i(jh) - \Delta\bar{r}(jh)|.\end{aligned} \tag{10}$$

Therefore

$$\begin{aligned}\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \lambda_2^{n+1}\sqrt{N}\kappa_0 + \\ + \sqrt{N}\sum_{j=0}^{n}\lambda_2^{n-j}\max_i|\Delta r_i(jh) - \Delta\bar{r}(jh)|,\end{aligned} \tag{11}$$

with $\kappa_0 = \max_i|r_i(-h) - \bar{r}(-h)|$.

From Assumption 2.3,

$$\max_i|\Delta r_i(jh) - \Delta\bar{r}(jh)| \leq h\theta, \tag{12}$$

which substituted in (11) implies that

$$\begin{aligned}&\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \\ &\leq \lambda_2^{n+1}\sqrt{N}\kappa_0 + \sqrt{N}h\theta\sum_{j=0}^{n}\lambda_2^{n-j}.\end{aligned} \tag{13}$$

Noting that $\sum_{j=0}^{n}\lambda_2^{n-j}$ is a geometric series of factor λ_2,

$$\begin{aligned}&\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \\ &\leq \lambda_2^{n+1}\sqrt{N}\kappa_0 + \sqrt{N}h\theta\frac{1-\lambda_2^n}{1-\lambda_2}.\end{aligned} \tag{14}$$

Taking the limit when n goes to infinity the first term goes to zero and consequently if $h \in (0, h^*]$, then

$$\lim_{n\to\infty}\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \delta. \tag{15}$$

■

Let us note that our bound reflects the dependence on the number of nodes and the network connectivity in a very simple way. Compared to (3), it does not contain powers of the number of nodes and is divided by the square root of N instead of N. In this sense, the following remark improves even more the bound on the step size.

***Corollary** 3.2:* Assume that the conditions in Theorem 3.1 hold and that $\|\Delta\mathbf{r}(nh) - \Delta\bar{r}(nh)\mathbf{1}\|_2 \leq h\bar{\theta}$ for all $h > 0$ and $n \geq 0$. Then the choice of

$$h^* = \frac{\delta(1-\lambda_2)}{\bar{\theta}} \tag{16}$$

achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ.

Proof: Replacing the new bound in equation (10) the result holds. ■

Regarding the algebraic connectivity, if the topology is very dense, then the nodes can communicate with many neighbors and λ_2 is close to 0, allowing larger values of h^*. On the other hand, if the network is very sparse, then λ_2 approaches to 1, which implies that h^* will be closer to zero. A more developed comparison of our bound with the original in (3) will be done in the next section with time-varying topologies.

IV. Step size for time-varying communication topologies

In this section, we resume the analysis with time-varying topologies, deriving a bound on the step size for the general case. After that we demonstrate that our bound is better than the one in (3) for any number of nodes or connectivity.

As in the previous version, we base our results on the eigenvalues of the weight matrices. First of all, let us define

$$\mathbf{\Pi}(n) = \prod_{k=0}^{B}\mathbf{A}(n+B-k). \tag{17}$$

By Assumption 2.1 we know that $\mathbf{\Pi}(n)$ is doubly stochastic for all n. Therefore, $\mathbf{\Pi}(n)$ can be considered as a different weight matrix associated to the communication graph [16] $\mathcal{G}(n) = (\mathcal{V}, \mathcal{E}(n) \cup \mathcal{E}(n+1) \cup \ldots \cup \mathcal{E}(n+B))$, which because of Assumption 2.2 we know that it is connected. Therefore, as happened with $\mathbf{A}$ in the previous section, $\mathbf{\Pi}(n)$ has one eigenvalue $\lambda_1 = 1$ and the rest eigenvalues satisfy $\lambda_1 > \lambda_2 > \ldots > \lambda_N > -1$.

Taking this into account we denote

$$\lambda_M = \sup_n \rho\left(\mathbf{\Pi}(n) - \frac{\mathbf{1}^T\mathbf{1}}{N}\right) < 1, \tag{18}$$

with $\rho(\cdot)$ representing the spectral radius operator, the supreme of the second largest eigenvalue of the different products of B consecutive weight matrices. Note that

due to Assumption 2.1 the supreme is not 1. In fact, using the Metropolis Weights it can be shown that $\lambda_M < \frac{1-2\cos(\frac{(N-1)\pi}{N})}{3}$, [17].

We can now formulate the result on the step size for the time-varying case:

***Theorem** 4.1 (Step size for time-varying topologies):* Let δ be a positive constant and

$$h^* = \frac{\delta(1-\lambda_M)}{B\theta\sqrt{N}}. \tag{19}$$

Under Assumptions 2.1, 2.2 and 2.3 the implementation of (1) with $h \in (0, h^*]$ and initial conditions $x_i(0) = r_i(-h),\ i \in (1,\dots,N)$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ.

Proof: Along the proof we let $\lfloor x \rfloor : \mathbb{N} \to \mathbb{N}$ be the operator that returns the closest integer upper-bounded by x, i.e., the floor operator.

Using the same argument as in equation (10), we know that after B iterations the Euclidean norm of the error is reduced by a factor λ_M. Thus

$$\begin{aligned}
&\|\prod_{k=0}^{n-j}\mathbf{A}(k)\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_2 \leq \\
&\qquad \leq \lambda_M^{\lfloor\frac{n-j}{B}\rfloor}\|\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_2 \leq \\
&\qquad \leq \lambda_M^{\lfloor\frac{n-j}{B}\rfloor}\sqrt{N}\|\Delta\mathbf{r}(jh) - \Delta\bar{r}(jh)\mathbf{1}\|_\infty = \\
&\qquad = \lambda_M^{\lfloor\frac{n-j}{B}\rfloor}\sqrt{N}\max_i|\Delta r_i(jh) - \Delta\bar{r}(jh)| \leq \\
&\qquad \leq \lambda_M^{\lfloor\frac{n-j}{B}\rfloor}\sqrt{N}h\theta,
\end{aligned}$$

where in the last step we have used Assumption 2.3.

Therefore

$$\begin{aligned}
&\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \\
&\quad \leq \lambda_M^{\lfloor\frac{n+1}{B}\rfloor}\sqrt{N}\kappa_0 + \sqrt{N}h\theta\sum_{j=0}^{n}\lambda_M^{\lfloor\frac{n-j}{B}\rfloor} \leq \\
&\quad \leq \lambda_M^{\lfloor\frac{n+1}{B}\rfloor}\sqrt{N}\kappa_0 + \sqrt{N}h\theta B\sum_{j=0}^{\lfloor\frac{n}{B}\rfloor}\lambda_M^{j} = \\
&\quad = \lambda_M^{\lfloor\frac{n+1}{B}\rfloor}\sqrt{N}\kappa_0 + \sqrt{N}h\theta B\frac{1-\lambda_M^{\lfloor\frac{n}{B}\rfloor}}{1-\lambda_M},
\end{aligned} \tag{20}$$

where $\kappa_0 = \max_i|r_i(-h) - \bar{r}(-h)|$. Taking the limit when n goes to infinity the first term goes to zero and consequently if $h \in (0, h^*]$, then

$$\lim_{n\to\infty}\|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \delta. \tag{21}$$

■

Note the similarity of the bound for fixed topologies with the bound for time-varying topologies. It turns out that the bound in the latter case can be expressed similarly as in the former case considering rounds of B updates. Now we demonstrate that, for a fixed δ (19) is always larger than (3).

***Lemma** 4.2:* The parameter α in Assumption 2.1 is upper bounded by $\alpha < 1/2$.

Proof: By considering a topology with a single link the result is immediate. ■

***Lemma** 4.3:* Let δ be a positive constant and a_{ij} defined according the Metropolis Weights [10]. For any $N > 1,\ B \geq 1,\ h^*$ in equation (19) is bigger than h_1 in equation (3).

Proof: The denominator in (19) is smaller than in (3) for all $N > 1$ and $B \geq 1$.

Regarding the numerators, using Lemma 4.2 we take the values $\alpha = 1/2$ and $B = 1$ to consider the biggest numerator in (3). If we use the Metropolis Weights, then $\lambda_M < \frac{1-2\cos(\frac{(N-1)\pi}{N})}{3}$, [17] in (19). Considering these two bounds it can be shown that the derivative of the numerator in (3) minus the numerator in (19) is negative. Noting that for $N = 2$, $\alpha^4 - (1-\lambda_M) < 0$ we conclude that the numerator in (19) is greater than in (3) and the result is proved for all N. ■

Therefore, for any desired steady-state error, δ, the step size h^* obtained by us is larger than the one obtained in [1]. Finally, let us remark that since we are considering exactly the same algorithm as in [1], those situations that were shown to reach average consensus for zero steady-state error are still valid:

***Corollary** 4.4 (Corollary 3.1 [1]):* Under Assumptions 2.1 and 2.2 if $\lim_{n\to\infty}\max_i\Delta r_i(nh) - \min_i\Delta r_i(nh) = 0$, the implementation of (1) with any $h > 0$ and initial state $x_i(0) = r_i(-h),\ i \in \{1,\dots,N\}$ achieves the dynamic average consensus with a zero steady-state error.

V. Step size for higher order consensus algorithms

In this section, we analyze the step size h, required to reach the average consensus when using higher-order algorithms. For simplicity we do the analysis considering a fixed and connected communication topology as in Section III, i.e., $\mathbf{A}(n) = \mathbf{A},\ \forall n \in \mathbb{N}$.

The general k^{th} order dynamic consensus is defined as

$$\begin{aligned}
\mathbf{x}^{[1]}(n+1) &= \mathbf{A}\mathbf{x}^{[1]}(n) + \Delta^{[k]}(nh), \\
\mathbf{x}^{[\ell]}(n+1) &= \mathbf{A}\mathbf{x}^{[\ell]}(n) + \mathbf{x}^{[\ell-1]}(n+1),
\end{aligned} \tag{22}$$

with

$$\Delta^{[\ell]}r_i(nh) = \Delta^{[\ell]}r_i(t) = \Delta^{[\ell-1]}r_i(t) - \Delta^{[\ell-1]}r_i(t-h), \tag{23}$$

$\ell \in \{1,\dots,k\}$ the ℓth order differences of the input signals.

***Assumption** 5.1: (Relatively bounded k^{th}-order differences)*: For any $h > 0$ there exist a time invariant constant θ_k such that

$$\max_i\Delta^{[k]}r_i(nh) - \min_i\Delta^{[k]}r_i(nh) \leq h\theta_k, \quad \forall n \geq 0.$$

Let us define the iterative variables

$$b_k(n) = \sum_{\ell=1}^{k} b_\ell(n-1), \quad k, n \in \mathbb{N}, \tag{24}$$

with $b_k(1) = 1$ for all k.

We will make use of the following two lemmas to find the bound on h:

***Lemma** 5.1:* The direct expression of $b_k(n)$ is equal to

$$b_k(n) = \frac{\prod_{\ell=0}^{k-2}(n+\ell)}{(k-1)!}, \tag{25}$$

for all $k, n > 1$.

Proof: We show it by induction on k and n. First of all, let us note that, according to (24), $b_1(n) = 1$ for all n. We show that (25) holds for any $k > 1$ and $n = 2$. Using (24)

$$b_k(2) = k = \frac{k!}{(k-1)!} = \frac{\prod_{\ell=0}^{k-2}(2+\ell)}{(k-1)!}. \tag{26}$$

Assume now that (25) is true for all n up to $k-1$, and for a certain $n > 1$ and any k. To see it also holds for $n+1$ and any k, observe

$$\begin{aligned} b_k(n+1) &= \sum_{\ell=1}^{k} b_\ell(n) = b_k(n) + b_{k-1}(n+1) = \\ &= \frac{\prod_{\ell=0}^{k-2}(n+\ell) + (k-1)\prod_{\ell=0}^{k-3}(n+1+\ell)}{(k-1)!} \\ &= \frac{(n-1+k-1)\prod_{\ell=0}^{k-3}(n+1+\ell)}{(k-1)!} \\ &= \frac{\prod_{\ell=0}^{k-2}(n+1+\ell)}{(k-1)!}. \end{aligned} \tag{27}$$

The induction is completed from the fact that the property is true for any k and $n = 2$. ■

***Lemma** 5.2:* The average of the inputs signals at iteration n, $\bar{r}(nh)$, is equal to:

$$\begin{aligned} \bar{r}(nh) = \bar{r}(-h) &+ \sum_{\ell=1}^{k-1} b_{\ell+1}(n)\Delta^{[\ell]}\bar{r}(-h) + \\ &+ \sum_{j=0}^{n} b_k(n-j+1)\Delta^{[k]}\bar{r}(jh), \end{aligned} \tag{28}$$

for all $k > 0$.

Proof: It is a consequence of using Lemma 5.1 and (23). ■

***Theorem** 5.3 (Step size for k^{th}-order fixed topologies):* Let δ be a positive constant and

$$h^* = \frac{\delta(1-\lambda_2)^k(k-1)!}{\theta_k\sqrt{N}}. \tag{29}$$

Under Assumptions 2.1 and 5.1 the implementation of (22) with $h \in (0, h^*]$ and initial conditions, $x_i^{[k-\ell]}(0) = \Delta^{[\ell]}r_i(-h), \ell = 1, \ldots, k-1$ and $x_i^{[k]}(0) = r_i(-h),\ i \in (1, \ldots, N)$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ.

Proof: Let us define

$$X(n) = (\mathbf{x}^{[k]}(n), \ldots, \mathbf{x}^{[k-1]}(n), \mathbf{x}^{[1]}(n))^T. \tag{30}$$

Developing (22), the discrete-time dynamics of the variable $X(n)$ is

$$X(n+1) = \begin{bmatrix} \mathbf{A} & \mathbf{A} & \ldots & \mathbf{A} \\ \mathbf{0} & \mathbf{A} & \ldots & \mathbf{A} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{A} \end{bmatrix} X(n) + \begin{bmatrix} \mathbf{I}_N \\ \mathbf{I}_N \\ \vdots \\ \mathbf{I}_N \end{bmatrix} \Delta^{[k]}\mathbf{r}(nh). \tag{31}$$

This leads to

$$\begin{aligned} \mathbf{x}^{[k]}(n+1) = &\sum_{\ell=1}^{k} b_\ell(n+1)\mathbf{A}^{n+1}\mathbf{x}^{[\ell]}(0) + \\ &+ \sum_{j=0}^{n} b_k(n-j+1)\mathbf{A}^{n-j}\Delta^{[k]}r(jh). \end{aligned} \tag{32}$$

Subtracting $\bar{r}(nh)\mathbf{1}$ and using Lemma 5.2 yields

$$\begin{aligned} &\mathbf{x}^{[k]}(n+1) - \bar{r}(nh)\mathbf{1} = \mathbf{A}^{n+1}\mathbf{r}(-h) - \bar{r}(-h)\mathbf{1} + \\ &+ \sum_{\ell=1}^{k-1} b_{\ell+1}(n+1)\left[\mathbf{A}^{n+1}\Delta^{[\ell]}\mathbf{r}(-h) - \Delta^{[\ell]}\bar{r}(-h)\mathbf{1}\right] + \\ &+ \sum_{j=0}^{n} b_k(n-j+1)\left[\mathbf{A}^{n-j}\Delta^{[k]}\mathbf{r}(jh) - \Delta^{[k]}\bar{r}(jh)\mathbf{1}\right]. \end{aligned} \tag{33}$$

Thus, by Lemma 5.1,

$$\begin{aligned} &\|\mathbf{x}^{[k]}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \\ &\leq \lambda_2^{n+1}\sqrt{N}\kappa_0 + \lambda_2^{n+1}\sqrt{N}\sum_{\ell=1}^{k-1} b_{\ell+1}(n+1)\kappa_\ell + \\ &+ \frac{h\theta_k\sqrt{N}}{(k-1)!}\sum_{j=0}^{n}\prod_{\ell=0}^{k-2}(n-j+\ell)\lambda_2^{n-j}, \end{aligned} \tag{34}$$

with $\kappa_0 = \max_i |r_i(-h) - \bar{r}(-h)|$ and $\kappa_\ell = \max_i |\Delta^{[\ell]}r_i(-h) - \bar{r}(-h)|$. The first two terms of the right hand side of the inequality approach to zero with n. Regarding the sum on the third term, note that

$$\frac{\partial^{k-1}\lambda_2^{n+k-1-j}}{\partial^{k-1}\lambda_2} = \prod_{\ell=0}^{k-2}(n+j+\ell)\lambda_2^{n-j}. \tag{35}$$

Therefore

$$\sum_{j=0}^{n}\prod_{\ell=0}^{k-2}(n-j+\ell)\lambda_2^{n-j} = \frac{\partial^{k-1}\frac{1-\lambda_2^n}{1-\lambda_2}}{\partial^{k-1}\lambda_2}, \tag{36}$$

which, in the limit, when $n \to \infty$, is equal to $1/(1-\lambda_2)^k$ and consequently if $h \in (0, h^*]$, then

$$\lim_{n\to\infty} \|\mathbf{x}(n+1) - \bar{r}(nh)\mathbf{1}\|_\infty \leq \delta. \tag{37}$$

■

***Remark** 5.1 (Infinitely large h):* Compared to the result in [1], let us note that, fixing δ and λ_2 in (29),

$$\lim_{k\to\infty} h^* = \infty.$$

This means that at some point, h^* is an increasing function of k, which implies that for any desired error δ, if the communication constraints do not allow a step size smaller than some h, we can always find an order such that for that step size the steady-state error is upper-bounded by δ. This is done at the expense of a higher computation cost.

Finally, although we do not prove it here, it can be shown that for time-varying topologies, the step size is bounded by

$$h^* = \frac{\delta(1-\lambda_M)^k(k-1)!}{B\theta_k\sqrt{N}}. \tag{38}$$

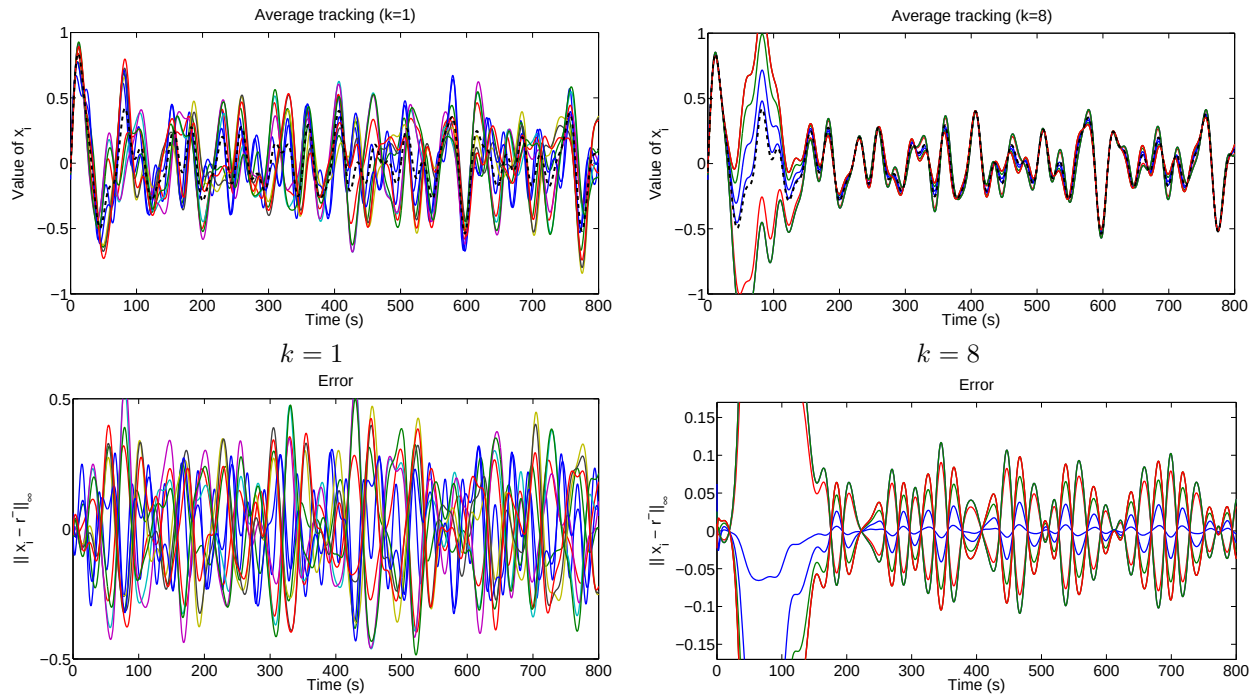

Fig. 2: Dynamic average consensus using two different order algorithms, $k = 1$ and $k = 8$. The top figures show the values of the 10 nodes when estimating the average of their signals with the average of $r_i(t)$ represented by the black dashed line. Bottom figure shows the error of all the nodes with respect to the average. Using the same step size, $h = 0.5$, the first order algorithm (left) does not achieve the consensus with the desired steady-state error whereas the algorithm with $k = 8$ is able to do it.

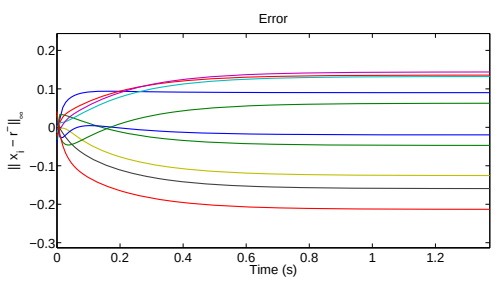

Fig. 1: Steady-state error tracking ramp inputs with a first order consensus algorithm using step size $h = 0.0091$.

VI. Simulations

In this section, we illustrate our results with some simulations. Firstly, we analyze the quality of our bound for the first order consensus algorithm compared to (3). We consider a fixed network composed by 10 nodes with algebraic connectivity $\lambda_2 = 0.91$. The input signals of the ith node is defined as $r_i(t) = it$, which implies that Corollary 3.2 can be applied with $\bar{\theta} = \|(1, \ldots, N)^T - (N+1)/2\mathbf{1}\|_2 = 9.0830$. Assigning $\delta = 1$ implies that $h^* \leq 0.0091$ in equation (16) whereas the maximum allowed step size considering (3) is equal to $h_1 < 9.6374 \cdot 10^{-20}$, i.e., the new bound is around 10^{17} times larger than the original one. In Fig. 1 we show the errors of the ten nodes with respect to the average which, as theorized, are upper bounded by $\delta = 1$. Additionally, note that this bound is pretty accurate, the maximum error in the figure is equal to $\|\mathbf{x}(n) - \bar{r}(nh)\mathbf{1}\|_\infty = 0.2132$, which implies that the maximum step size that guarantees $\delta \leq 1$ is only $\delta / \|\mathbf{x}(n) - \bar{r}(nh)\mathbf{1}\|_\infty = 4.6913$ times larger than the one in our results.

The previous toy example served just to show the improvement in the step size (considering $k \geq 2$ the algorithm reaches dynamic average consensus with zero steady-state error). Let us consider now a more interesting example with the same network topology. In this example the communications between nodes are constrained to be lower bounded by half a second, i.e., $h \geq 0.5$. The signals measured by the nodes are

$$r_i(t) = \sin(\frac{0.25}{\sqrt{i}} t),$$

and the desired steady-state error is equal to $\delta = 0.12$. Noting that Assumption 5.1 holds with $\theta_k \leq 2h^{k-1} 0.25^k$, we realize that for $k = 1$ the maximum allowed step size is equal to $h^* = 0.0063$. On the other hand, by increasing k we obtain larger and larger step sizes, reaching that for $k = 8$ the constraints of the problem are satisfied. In Fig. 2 we depict value of the nodes and the error for the two cases. Left figures show the execution of the algorithm with $k = 1$ and $h = 0.5$ and right figures show the execution with $k = 8$ and $h = 0.5$. As we can see, in the second scenario there is an initial phase where the error grows a lot (the transient phase). However, after some point the nodes are able to track the average of the signals with error bounded by the desired constant.

VII. Conclusions

In this paper, we have presented a different study on the discrete-time dynamic average consensus algorithms initially proposed in [1]. Using the information about the eigenvalues of the weight matrices, we have come up with a different demonstration about the maximum step size that the nodes can choose. With our results, the step size allowed to the

nodes is always bigger than the one initially found, independently of the number of nodes or the network connectivity. Examples show the difference can be of several orders of magnitude. We have also shown there is always a kth order dynamic consensus algorithm leading to an arbitrarily small steady-state error with an arbitrarily large step size. Our future work will focus on using acceleration techniques to improve the convergence rates.

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