Distributed and robust resource allocation algorithms for multi-agent systems via discrete-time iterations

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***Abstract*— This paper proposes two novel nonlinear discrete-time distributed algorithms to solve a class of resource allocation problems. The proposed algorithms allow an interconnected group of agents to collectively minimize a global cost function subject to equality and inequality constraints. Under some technical conditions, we show that the algorithms converge to the solution in a practical way as long as the stepsize chosen is sufficiently small. Of particular interest is that the proposed algorithms are designed to be robust so that temporary errors in communication or computation do not change their convergence to a neighborhood around the equilibrium, and to this end, agents do not require global knowledge of total resources in the network or any specific procedure for initialization. The convergence of the algorithms is established via second-order convexity theory together with nonsmooth Lyapunov analysis. To illustrate the applicability of our strategies, we study a virus mitigation problem over computer and human networks.**

I. Introduction

Distributed resource allocation is a general problem in which a group of agents decides how to assign a set of scarce resources to solve a common objective while satisfying operational and communication constraints. The problem is often formulated via an objective function in the form of a weighted sum of individual costs, which models a fair agent contribution towards the reduction of the shared cost.

A specific real-world problem leading to such a setting arises in computer networks, epidemiology, and viral marketing, where a viral outbreak can threat to the security of interconnected infrastructure and the well-being of the general public. The implementation of strategies to stop epidemics can be specially challenging when networks are operated by multiple managers who need to preserve the privacy and interests of their local users. These scenarios would benefit from the development of distributed anonymous coordination algorithms that allow the implementation of best responses in a robust way, where it is not required a specific initialization and the convergence is not affected by a single erroneous update in the system state. For this reason, a distributed and robust implementation of these optimal responses over networks calls for the use of distributed algorithms that the multiple operators can employ for this purpose. Motivated by this problem, we propose distributed and robust algorithms for resource allocation, which converge regardless of the initial condition under some technical assumptions.

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Literature review. In a distributed resource optimization problem, the cost function may be linear or nonlinear. The linear scenario has been extensively studied in the control literature. Some approaches utilize consensus-based algorithms [1] and gossip-based interactions [2]. For consensus-based algorithms, robust solutions can be always obtained by using ideas from [3], where consensus is achieved independently what the initial states are. On the other hand, when the utility function is nonlinear, some approaches are based on dual decomposition methods, e.g., [4] for unconstrained problems, or [5], [6] for constrained ones. Other approaches are based on a combination of subgradients and consensus [7], or on the local version of the replicator equation [8], [9], and yet others are based on gossip algorithms [10], or saddle-point methods [11], or Laplacian gradient dynamics [12], [13]. However, such approaches are not proven to be robust since they assume no errors in communication or computations. In addition, the total amount of resources available to all agents needs to be known in advance in order to initialize the algorithm.

Statement of contributions. Here, we present and analyze two novel distributed discrete-time nonlinear algorithms to solve a class of distributed resource allocation problems. In particular, we extend our previous work [14] by providing alternative discrete-time algorithms with provable convergence guarantees to the solution of a class of resource allocation problems. Our approach allows an interconnected group of agents to collectively minimize a global cost function subject to equality and inequality constraints. Under some technical conditions, we show that the algorithms converge to the solution in a practical way as long as the chosen stepsize is sufficiently small. Of particular interest is that the proposed algorithms are convergent to a neighborhood around the equilibrium even when there are temporary errors in communication or computation. And thus, agents do not require global knowledge of total resources in the network or any specific procedure for initialization. We analyze the algorithms over weight-balanced and strongly connected networks. Finally, we illustrate the applicability of our algorithms on a virus spreading problem over computer and human networks. In this application we approximate the gradient of the cost function by means of the well-known distributed power iteration algorithm.

II. Preliminaries

This section presents notation and basic notions from graph, matrix, and stability theory that are used in the sequel.

A. Notation and graph-theoretic notions

We denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{R}^N_{>0}$ the positive orthant of $\mathbb{R}^N$, for some $N \in \mathbb{N}$, I_N the identity matrix of size $N \times N$, $\mathrm{diag}(a_1, \ldots, a_N)$ the $N \times N$ matrix with entries a_i along the diagonal, and $\mathbf{1}_N \in \mathbb{R}^N$ the column vector whose elements are all equal to one. The spectrum of A is denoted by $\mathrm{spec}(A)$, an eigenvalue of a symmetric matrix $A \in \mathbb{R}^{N\times N}$ is denoted by $\lambda_i(A)$, where $\lambda_1(A) \geq \ldots \geq \lambda_N(A) \in \mathrm{spec}(A)$ and the singular values are denoted $\sigma_1(A) \geq \ldots \geq \sigma_N(A)$. When we use inequalities for vectors, we refer to componentwise inequalities. We let $[l]_+ = \max\{0, l\}$, for $l \in \mathbb{R}$. The two-norm and ∞-norm of a vector are denoted by $\|.\|_2$ and $\|.\|_\infty$, respectively.

A matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}_{\geq 0}$ is called *nonnegative* if $a_{ij} \geq 0$, for all $i, j \in \{1, \ldots, N\}$. A directed graph of order N or *digraph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$, the *vertex set*, is a set with N nodes, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, the *edge set*, is a set of ordered pair of vertices called edges. Given $B \in \mathbb{R}^{N\times N}_{\geq 0}$, its associated *weighted digraph* $\mathcal{G}(B)$ is the graph with $\mathcal{V} = \{1, \ldots, N\}$ and edge set defined by the following relationship: $(i, j) \in \mathcal{E}(B)$ if and only if $b_{ij} > 0$. The associated weight of the edge (i, j) is given by the entry b_{ij}. The digraph $\mathcal{G}(B)$ is said to be *weight-balanced* if $\sum_{j=1}^N b_{ij} = \sum_{j=1}^N b_{ji}$ for all $i \in \mathcal{V}$. Given a pair of indices $i, j \in \mathcal{V}$ of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, j is called an *out neighbor* of i if $(i, j) \in \mathcal{E}$. We let $\mathcal{N}_i^{\mathrm{out}}(\mathcal{G})$ denote the set of out neighbors of i in $\mathcal{G}$. A digraph $\mathcal{G}(A)$ is *strongly connected* if there exists a path between any two vertices. The strongly connectedness of $\mathcal{G}(A)$ is equivalent to requiring that A is an *irreducible matrix*. The *Laplacian matrix* associated to a digraph $\mathcal{G}(A)$ is defined as $L(\mathcal{G})_{ii} = \sum_{j=1}^N a_{ij}$, and $L(\mathcal{G})_{ij} = -a_{ij}$ for $i \neq j$.

B. Partial stability for nonsmooth Lyapunov functions

The notions we introduce here follow [15], [16]. Given two sets S and T, a *set-valued map*, denoted by $h : S \rightrightarrows T$, associates to an element of S a subset of T. Consider a discrete-time dynamical system given by the difference inclusion in $\mathbb{R}^n$

$$x(t+1) \in \mathcal{H}(x(t), w(t)), \tag{1}$$

where $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a set-valued map for some $n, m \in \mathbb{N}$, x is the state, w is the input, and $t \geq 0$. We assume that $\mathcal{H}$ assigns to each point $(x, w) \in \mathbb{R}^n \times \mathbb{R}^m$ a nonempty set $\mathcal{H}(x, w) \subset \mathbb{R}^n$. Consider the unforced system (1), i.e., $w = 0$. We divide the the state x in two components: (1) the y-component used to study the stability of the equilibrium $x^* = 0$, and (2) other (non-controlled) z-component, so that $x = [y^\top, z^\top]^\top$. We use $x(t) = x(t; t_0, x_0)$ to denote the solution of the system (1) given the initial condition x_0 at t_0. When $w = 0$, we have the following definition.

Definition 1: Let D_φ be a domain of initial conditions x_0 such that $\|y_0\|_2 < \varphi$ and $\|z_0\|_2 \leq L$, where $L \in \mathbb{R}_{>0}$. The origin of system (1) is said to be

a y*-stable* if for any $\varepsilon > 0$, $t_0 \geq 0$, one can find $\varphi(\varepsilon, L) > 0$ such that $x_0 \in D_\varphi$ yields $\|y(t; t_0, x_0)\|_2 < \varepsilon$ for all $t \geq t_0$.

b *globally asymptotically* y*-stable* if it is y-stable, and an arbitrary solution $x(t)$ of the system (1) exists for all $t \geq 0$, $y_0 \in K_y$, where K_y is an arbitrarily compact set in y-space, and $x(t)$ is y-bounded and satisfies $\lim \|y(t; t_0, x_0)\|_2 = 0$ as $t \to +\infty$.

The smooth version of the next lemma was introduced in [15], Theorem 2. We present here an adaptation to nonsmooth Lyapunov functions. The proof is straightforward when considering the analogous definitions from nonsmooth analysis and follows the same steps as in [15].

Lemma 1: Suppose that there exists a continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$, and a continuous vector function $W : \mathbb{R}^n \to \mathbb{R}^n$ for the unforced system (1) in $\mathbb{R}^n$ with $W(0) = 0$. Then, the origin is globally asymptotically y-stable if the following conditions are met

$$V(0, z) = 0, \tag{2a}$$
$$\alpha_1(\|y\|_2) \leq V(x) \leq \alpha_2(\|\vartheta\|_2), \tag{2b}$$
$$\sup_{g \in \mathcal{H}(x)} V(g) - V(x) \leq -\alpha_3(\|\vartheta\|_2), \tag{2c}$$

where $\vartheta = [y^\top, W(x)^\top]^\top$, α_1 belongs to class $\mathcal{K}_\infty$, α_2, α_3 belong to class $\mathcal{K}$.

Next definition is an adaptation of *input-to-output stability* (e.g., see [17]) when we consider the output of the system (1) as the controllable states y.

Definition 2: (y-input-to-state stability): The system (1) is said to be y-input-to-state stable (for short y-ISS) if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that for any initial state $y(t_0)$ and any bounded input $w(t)$, the solution $y(t)$ for $t \geq 0$ satisfies

$$\|y(t)\| \leq \beta(\|\vartheta_0\|, t - t_0) + \gamma\Big(\sup_{t_0 \leq \tau \leq t} w(\tau)\Big)$$

Theorem 1: Suppose that there exists a continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0, z) = 0, \tag{3a}$$
$$\alpha_1(\|y\|) \leq V(x) \leq \alpha_2(\|\vartheta\|), \tag{3b}$$
$$\sup_{g \in \mathcal{H}(x)} V(g) - V(x) \leq -\alpha_3(\|y\|), \; \forall \, \|y\| \geq \rho(\|w\|) \tag{3c}$$

$\forall (x, w) \in \mathbb{R}^n \times \mathbb{R}^m$, where $\vartheta = [y^\top, W(x)^\top]^\top$, α_1 belongs to class $\mathcal{K}_\infty$, α_2, α_3, ρ belong to class $\mathcal{K}$, and a continuous vector function $W : \mathbb{R}^n \to \mathbb{R}^n$ for the system (1) with $W(0) = 0$. Then the system (1) is *y-input-to-state stable* with $\gamma = \alpha_1^{-1}(\alpha_2(\rho))$.

III. Problem statement, solution approach, and algorithms

In this section, we introduce the optimization problem we are set out to solve, which is followed by the proposed ROBUST GRADIENT FAIRNESS and ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithms with guaranteed convergence to their corresponding optimizer under complementary sets of assumptions.

A. Problem statement and solution approach

We consider a network of N agents connected over a digraph whose goal is to minimize a general $f : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ under resource constraints. The BOX-COUPLED FAIRNESS optimization problem is given by

$$\begin{aligned} \min_{p} \ & f(p) \\ \text{s.t.} \ & \mathbf{1}_N^\top p = \mathbf{1}_N^\top u, \\ & p \in [\underline{p}, \overline{p}]^N, \end{aligned} \tag{4}$$

where f is the payoff, $p = [p_1, \dots, p_N]^\top \in \mathbb{R}^N$ is the resource allocation, $u_i \in \mathbb{R}$ is the input assumed to be constant that represents the available quantity of resources for each agent, $u = [u_1, \dots, u_N]^\top$, and $\underline{p}, \overline{p} \in \mathbb{R}^N$ are the lower and upper limits of the optimization variable, respectively. We name the last constraint in (4) as the *box constraint*. We simply refer to the problem with the box constraint omitted as the LINEARLY COUPLED FAIRNESS optimization problem. To solve both problems we state the following assumption.

Assumption 1: (Problem assumptions): We assume that the set of optimal solutions of (4) is nonempty, the payoff $f(p)$ is twice continuously differentiable, strongly convex with $\gamma I \leq \nabla_p^2 f(p) \leq \Gamma I$ for $\gamma, \Gamma \in \mathbb{R}_{>0}$, and bounded below. Also, an agent $i \in \mathcal{V}$ should be able to compute $\frac{\partial f}{\partial p_i}$ using only local information from $\mathcal{N}_i^{\text{out}}$ and $\|\nabla_p f(p)\| \leq M$. Under the same assumptions as for the last problem and using the exact penalty method (see, e.g., [18]), we reformulate the BOX-COUPLED FAIRNESS problem as follows:

$$\begin{aligned} \min_{p} \ & \hat{f}(p) \\ \text{s.t.} \ & \mathbf{1}_N^\top p = \mathbf{1}_N^\top u, \end{aligned} \tag{5}$$

where $\hat{f}(p) \triangleq f(p) + J(p)$, $J(p) \triangleq \epsilon \sum_{i=1}^N \Big([\underline{p_i} - p_i]_+ + [p_i - \overline{p_i}]_+ \Big)$, and $\epsilon \in \mathbb{R}_{>0}$. In what follows we use the following notation. We refer to $\mathcal{F}_{\leq}^u \triangleq \{p \in \mathbb{R}^N \mid \mathbf{1}_N^\top p \leq \mathbf{1}_N^\top u\}$, $\mathcal{F}_{\geq}^u \triangleq \{p \in \mathbb{R}^N \mid \mathbf{1}_N^\top p \geq \mathbf{1}_N^\top u\}$, $\mathcal{F}^u \triangleq \mathcal{F}_{\leq}^u \cap \mathcal{F}_{\geq}^u$, and $\mathcal{F}_{\text{box}}^{\nu} = \{p \in \mathbb{R}^N \mid \underline{p} - \nu \mathbf{1}_N \leq p \leq \overline{p} + \nu \mathbf{1}_N\}$ for $\nu \in \mathbb{R}_{>0}$. Under the assumptions we have laid out above, the next lemma characterizes the optimal solution to the BOX-COUPLED FAIRNESS optimization problem. Next lemma is a result from applying the exact penalty method and the characterization in [12] to the above problem.

Lemma 2: (Solution of the BOX-COUPLED FAIRNESS *problem):* Let Assumption 1, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\epsilon > 2 \max_{p \in \mathcal{F}^u} \|\nabla_p f(p)\|_\infty. \tag{6}$$

Then, the solution to the BOX-COUPLED FAIRNESS optimization problem satisfies

$$\zeta^* \mathbf{1}_N \in \nabla_p f(p) + \partial J(p), \tag{7a}$$

$$\mathbf{1}_N^\top p^* = \mathbf{1}_N^\top u, \tag{7b}$$

where $\zeta^* \in \mathbb{R}$ is the Lagrange multiplier for the equality constraint of the BOX-COUPLED FAIRNESS problem.

Next, we propose two distributed discrete-time algorithms which successfully converge to the solutions of the BOX-COUPLED FAIRNESS and LINEARLY COUPLED FAIRNESS problems introduced above under the corresponding assumptions. We will refer to them as the ROBUST GRADIENT FAIRNESS and ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithms.

B. Proposed algorithms

In order to solve the LINEARLY COUPLED FAIRNESS problem dynamically, we introduce the following ROBUST GRADIENT FAIRNESS algorithm,

$$w^+ = w - \alpha L \nabla_p f(p), \tag{8a}$$

$$p^+ = p + \alpha(-L^2 \nabla_p f(p) + Lw - p + u), \tag{8b}$$

where $w \in \mathbb{R}^N$ is an internal estimator state, $\alpha \in (0, 1)$ is the step size, and L is the Laplacian matrix associated to directed graph $\mathcal{G}$.

Since the cost function of the BOX-COUPLED FAIRNESS problem is assumed to be nonsmooth but convex, the previous algorithm can be adapted as the following ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm to solve the BOX-COUPLED FAIRNESS problem

$$w^+ \in w + \alpha(\xi_{\max} \mathbf{1}_N - \xi) \tag{9a}$$

$$p^+ \in p + \alpha(-L\xi + Lw - p + u), \tag{9b}$$

where w, α, and L have the same meaning as in the previous algorithm, $\xi_{\max} = \{\xi_i \in (\partial \hat{f}(p))_i \mid i = \operatorname{argmax}_{l \in \mathcal{V}} \max \xi_l\}$, and $\hat{f}$ has the same meaning as in (5). Notice that $\hat{f}$ is convex, locally Lipschitz, with generalized gradient $\partial \hat{f}(p) : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ given by $\partial \hat{f}(p) = \nabla_p f(p) + \partial J(p)$. Then ξ above is an element in the generalized gradient of $\hat{f}$.

IV. STABILITY ANALYSIS

In this section, we show that the equilibrium points of the ROBUST GRADIENT FAIRNESS and ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS dynamics coincide with the optimal solutions of the corresponding problems they solve, respectively, under the stated assumptions when $\mathcal{G}$ is strongly connected and weight-balanced. Theorem 2 and Theorem 3 present the stability properties of both dynamics. All results of this paper will be found in a forthcoming publication.

Lemma 3: (Equilibria of the ROBUST GRADIENT FAIRNESS *algorithm):* Let Assumption 1, on the payoff characteristics for the LINEARLY COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be a weight-balanced and strongly connected graph. Then, the ROBUST GRADIENT FAIRNESS algorithm has a unique solution p^* to

$$\mathbf{1}_N^\top p = \mathbf{1}_N^\top u, \tag{10a}$$

$$(\nabla_p f(p))_i = (\nabla_p f(p))_j, \ \forall i, j \in \{1, \dots, N\}, \tag{10b}$$

Lemma 4: (Equilibria of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS *algorithm):* Let Assumption 1, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be a weight-balanced and strongly connected graph. A point p^*

is a unique solution of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm iff there exists $\gamma^* \in \mathbb{R}$ such that

$$\gamma^* \mathbf{1}_N \in \partial \hat{f}(p^*), \tag{11a}$$
$$\mathbf{1}_N^\top p^* = \mathbf{1}_N^\top u. \tag{11b}$$

Before presenting our main results, the next lemma characterizes the invariance of $\mathcal{F}^u_{\leq}$ and $\mathcal{F}^u_{\geq}$ with respect to the ROBUST GRADIENT FAIRNESS dynamics. The same is true for the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS dynamics.

Lemma 5: (Invariance of the resource constraint under (8)*):* Let Assumption 1, on the payoff characteristics for the LINEARLY COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be a weight-balanced and strongly connected graph. Assume $\alpha \in (0,1)$ in (8). Then, the sets $\mathcal{F}^u_{\leq}$ and $\mathcal{F}^u_{\geq}$ are strongly positively invariant under the ROBUST GRADIENT FAIRNESS dynamics.

Theorem 2: (Sufficient conditions for convergence of the ROBUST GRADIENT FAIRNESS *algorithm):* Let Assumption 1, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Assume f is radially unbounded. Let $\mathcal{G}$ be a weight-balanced and strongly connected graph. For any constant input $u \in \mathbb{R}^N$ and any initial state $p(0)$, $w(0)$, the solutions of the system (8) converge asymptotically to the equilibrium point (10) if $\alpha \in (0, \min\{1, \frac{\lambda_2(L^2+L^{2\top})}{2\Gamma\sigma_1^2(L^2)}\})$.

Before presenting our main result in Theorem 3, we show next supporting lemma, where it is shown the solutions of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS dynamics are bounded.

Lemma 6: (Boundedness of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS *dynamics):* Let Assumption 1, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be weight-balanced and strongly connected. Then, the set $\mathcal{F}^\nu_{\text{box}}$ is strongly positively invariant under the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm provided that $\epsilon \in \mathbb{R}$ satisfies

$$\epsilon > \frac{1}{\min_{(i,j)\in\mathcal{E}} a_{ij}} \Big(2 d_{\text{out,max}} \max_p \|\nabla_p h(p)\|_\infty + \|Lw(0) - p(0) + u\|_\infty \Big), \tag{12}$$

where $d_{\text{out,max}} = \max_{i\in\mathcal{V}} \sum_{j=1}^N a_{ij}$.

Theorem 3: (partial-ISS of the BOX-COUPLED FAIRNESS *algorithm):* Let Assumption 1, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Assume $\mathcal{G}$ be a weight-balanced and strongly connected graph. Assume that the BOX-COUPLED FAIRNESS algorithm has access to an approximation of the gradient in the form $\nabla_p f(p) + e$, where $e \in \mathbb{R}^N$ is the error term for the approximation. Assume that e is uniformly bounded, i.e., $\|e(t)\|_2 \leq \alpha K$ for some $K \in \mathbb{R}_{>0}$. Then, for any constant input $u \in \mathbb{R}^N$ and any initial state $p(0)$, $w(0)$, the solutions of the system (9) converge asymptotically to a ball centered at the equilibrium point (11) with radius dependent on α.

Remark 1: Our motivation to include the error term e in Theorem 3 is that in many applications the gradient of the payoff function only can be approximated. For example, in the following section, we show an application to the virus spread minimization, where the gradient is approximated by the well-known Power Iteration.

V. APPLICATION TO VIRUS SPREAD MINIMIZATION

As an application, we employ the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm on a virus spread minimization problem. Notice that if we remove the box constraints, we can similarly use the ROBUST GRADIENT FAIRNESS algorithm to solve the same problem. In this section we introduce the δ-VIRUS MITIGATION and κ-VIRUS MITIGATION problems, which we solve using the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm together with the exact penalty method as shown in Section III. Since our distributed algorithm approach requires the computation of the gradient, and the fact that the gradient for the δ-VIRUS MITIGATION and κ-VIRUS MITIGATION problems is not naturally distributed, we approximate it by using the well-known Power Iteration.

A. Problem statement and solution approach

The SIS (susceptible-infected-susceptible) model for virus dynamics proposed in [19] is given by

$$q_i(t+1) = \Big(1 - \prod_{j=1}^N (1 - a_{ij} q_j(t))\Big), \tag{13}$$

where $q_i(t) \in \mathbb{R}$ is the probability that node i is infected at time t, $i \in \{1, \dots, N\}$ and a_{ij} is defined as

$$a_{ij} = \begin{cases} \kappa_i \beta_{ij}, & \text{for } j \neq i, \\ 1 - c_i \delta_i, & \text{for } j = i. \end{cases}$$

Here, $\kappa_i \in (0,1]$ represents the scaling factor of the nominal weight β_{ij}, $\beta_{ij} \in [0,1]$ is the probability that the virus from node i infects node j, or in other words, it represents the isolation capability placed in the entering branches, $c_i \in [0,1]$ represents the district-specific scaling factor, and $\delta_i \in [0,1]$ is the probability of an infected node i to be recovered. Using the Weierstrass product inequality, valid for $a_{ij} q_j(t) \in [0,1]$, we obtain the following upper bound

$$q_i(t+1) \leq \sum_{j=1}^N a_{ij} q_j(t), \quad \forall i \in \{1, \dots, N\},$$

where $q(t) = [q_1(t), \dots, q_N(t)]^\top$. The previous inequality reads in vector notation as

$$q(t+1) \leq A(\delta,\kappa) q(t), \tag{14}$$

where $A(\delta,\kappa) \in \mathbb{R}^{N\times N}$ is defined as

$$A(\delta,\kappa) = \begin{bmatrix} 1 - c_1\delta_1 & \kappa_1\beta_{12} & \dots & \kappa_1\beta_{1N} \\ \kappa_2\beta_{21} & 1 - c_2\delta_2 & \dots & \kappa_2\beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_N\beta_{N1} & \kappa_N\beta_{N2} & \dots & 1 - c_N\delta_N \end{bmatrix} = I_N - D + KG. \tag{15}$$

Here, $A(\delta,\kappa)$, $D = \operatorname{diag}(c)\operatorname{diag}(\delta)$, $K = \operatorname{diag}(\kappa)$, and $G = A(0,\mathbf{1}_N) - I_N$. Let $\mathcal{G}(A(0,\mathbf{1}_N)) = \mathcal{G}(G)$ be the graph associated to the virus dynamics contact network. We define the *topology matrix* of the network as the matrix G. When there is no confusion, we will denote $\mathcal{G}(G)$ by $\mathcal{G}$. Next proposition shows that the dominant eigenvalue, $\lambda_1(A)$, governs the growth/decay rate of infection.

Proposition 1 ([19]): An epidemic described by (13) becomes extinct if and only if $\lambda_1(A) < 1$. Moreover, when an epidemic is diminishing, the probability of infection decays at least exponentially over time.

Inspired by [14], [20], we consider the following two problems to minimize the effects of virus contagion. The δ-VIRUS MITIGATION problem is defined by

$$\min_{\delta \in [\underline{\delta}, \overline{\delta}]^N} \lambda_1(A(\delta, \kappa)), \quad \text{s.t.} \quad \mathbf{1}_N^\top \delta = \mathbf{1}_N^\top u_\delta, \tag{16}$$

where κ is fixed, $\mathbf{1}_N^\top u_\delta$ is the total amount of antivirus available, and the constants $\underline{\delta_i}, \overline{\delta_i} \in [0, 1]$. The κ-VIRUS MITIGATION problem is given

$$\min_{\kappa \in [\underline{\kappa}, \overline{\kappa}]^N} \lambda_1(A(\delta, \kappa)), \quad \text{s.t.} \quad \mathbf{1}_N^\top \kappa = \mathbf{1}_N^\top u_\kappa, \tag{17}$$

where δ is fixed, $\mathbf{1}_N^\top u_\kappa$ is the total amount of isolation resources, and the constants $\underline{\kappa_i}, \overline{\kappa_i} \in (0, 1]$.

Remark 2: We refer to δ^* and κ^* to the solutions of Problem (16) or Problem (17), respectively. These solutions minimize the exponential decay/growth rate of (13) subject to resource constraints. Moreover, if the solutions δ^* or κ^* make the dominant eigenvalue $\lambda_1(A(\delta^*, \kappa))$ or $\lambda_1(A(\delta, \kappa^*))$ (for fix κ or δ depending on the problem) strictly less than one, then it is guaranteed that the disease free equilibrium, $q^* = 0$, is globally exponentially stable.

To analyze the solution of the δ-VIRUS MITIGATION and κ-VIRUS MITIGATION problems, we show in the following two lemmas that $\lambda_1(A)$ is a convex function of δ, and κ, respectively. Finally, in Lemma 9 we explicitly calculate the gradient of $\lambda_1(A)$ in order to use the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm using the exact penalty method presented in Section III.

Lemma 7 ([21]): Let B be nonnegative, and $C = \operatorname{diag}(\delta_1, \ldots, \delta_N)$. Then, the maximum eigenvalue of $B+C$, $\lambda_1(B + C)$, is a convex function of C.

Lemma 8: ([22] Convexity of $\lambda_1(KG)$): Let G be positive semidefinite, and $K = \operatorname{diag}(\kappa_1, \ldots, \kappa_N)$. Assume that $\kappa_i > 0$ for all $i \in \{1, \ldots, N\}$. Then, λ_1 of KG is a convex function of K.

Remark 3: Notice that in general $\operatorname{trace}(G) = 0$ for our virus application, which means that G is indefinite. This fact makes the κ-VIRUS MITIGATION problem to be non-convex, even when G is symmetric. We refer to [22] for further discussion about the convexity of $\lambda_1(KG)$. To approximate the solution of the κ-VIRUS MITIGATION problem, we can use the inequality $\lambda_1(KG) \leq (\lambda_1(K^2G^2))^{\frac{1}{2}}$ [23]. When G is symmetric, G^2 is positive semidefinite, then $\lambda_1(K^2G^2)$ is convex by Lemma 8. Therefore, we can obtain an upper-bound of the solution of the κ-VIRUS MITIGATION problem. The analysis of the error by using this approach is out of scope of this paper.

Lemma 7 and Lemma 8 show that $\lambda_1(A)$ is a convex function with respect to its arguments under some technical assumptions. Then we can aim to apply the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm on the resource allocation problem associated to this function. As shown in Lemma 2, information of the gradient of $\lambda_1(A)$ is required in order to evaluate if a solution is optimal and to implement our algorithm. For that reason, in the following lemma we provide the analysis to obtain such a gradient.

Lemma 9: Let v, and s be the left and right eigenvectors of the matrix $A(\delta, \kappa)$ as defined in (15). Then

$$\frac{\partial \lambda_1(A(\delta, \kappa))}{\partial \delta_i} = -c_i \frac{v_i s_i}{v^\top s} \tag{18}$$

and,

$$\frac{\partial \lambda_1(A(\delta, \kappa))}{\partial \kappa_i} = \frac{v_i}{v^\top s} \sum_{j \neq i} \beta_{ij} s_j \tag{19}$$

B. The Power Iteration

The power method is a well-known algorithm for approximating $\lambda_1(A)$ for $A \in \mathbb{R}^{N \times N}$. For a detailed description of this method the reader may consult [24], [25]. In this paper we restrict our discussion for A being primitive with $z(0) > 0$. In Remark 4 we explain how to relax the condition of primitivity to Mezler and irreducible matrices. The algorithm is given by

$$z(t+1) = \frac{Az(t)}{\|Az(t)\|_\infty}, \tag{20}$$

where $z(t) \in (0, 1]$ for $t \in \mathbb{N}$. Under the assumptions listed above on A, $z(t) \to x$ as $t \to +\infty$, where x is the right eigenvector associated to $\lambda_1(A)$.

Remark 4: In general, the condition A to be primitive can be relaxed to have A nonnegative and irreducible. For that, it can be used a shifted version $A_c \triangleq A + cI$, where $c > 0$.

C. Algorithm Implementation and Simulations

Next we show an example that illustrates the response of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm to solve a particular δ-VIRUS MITIGATION problem. Before that, we summarize the required assumptions to solve the δ-VIRUS MITIGATION and the κ-VIRUS MITIGATION problems by using Theorem 3. First, Lemma 7 shows that $\lambda_1(A)$ is a convex function respect δ. Our assumptions require the problem payoff to be strongly convex. However, it's been observed in many example problems that this assumption is more restrictive than necessary. Because of this, we are currently studying how to relax proofs to be able to relax these conditions. For the case of $\lambda_1(A)$, Example 1 shows that for a particular problem, the algorithm converges to the desired solution. Second, we require to have $\lambda_1(A)$ bounded below and $A(\delta, \kappa)$ has to be irreducible all time. These assumptions are satisfied since the set $\mathcal{F}^\nu_{\text{box}}$ is invariant to our dynamics as shown in Lemma 6. Finally, the computation of the gradient of $\lambda_1(A)$ is required by using local information. To address this, we use the Power

Iteration method, summarized in Section V-B, to approximate the gradient as shown in Lemma (9). At each time step, the Power Iteration algorithm runs until a desired stopping condition is reached. The reader may consult [25] for a discussion on the stopping criteria.

Example 1 (Optimizing in δ): We illustrate the response of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm for the undirected topology matrix G associated to $A(\delta, \kappa)$ for fix κ. We construct G as a ring with $\mathcal{V} \in \{1, \dots, 10\}$, bidirectional edges given by $(i, i+1) = 1/5$ for $i \in \mathcal{V}$ (assume that if $i = 10$, then $i + 1 = 1$) and additional bidirectional edges given by $(1, 5) = (3, 9) = 1/6$. We use $u_1 = 5.5$, $u_j = 0$ for $j \in \mathcal{V} - \{1\}$, $c_1 = c_3 = c_6 = 0.85$, $c_j = 1$ for $j \in \mathcal{V} - \{1, 3, 6\}$, $\overline{\delta} = .9\mathbf{1}_N$, $\underline{\delta} = .2\mathbf{1}_N$, $\epsilon = 8.47$, and $\alpha = 0.01$. In this example, we approximate the gradient of $\lambda_1(A)$ by the Power Iteration. At each iteration of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm, we run one iteration of the Power Iteration. In Figure 1, we show the behavior of the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm for a random initial condition with $\delta(0) \in [.2, .9]^N$, $w(0) \in [0, 1]^N$. The optimal value is given by $\lambda_1(A(\delta^*)) = 0.9455$. We introduce an erroneous update on the system state at time $t = 1500$, where we force $p(1500)$ and $w(1500)$ to a random vector in $[0, 1]^N$. After $t = 1500$, the algorithm converges again to the optimal point. Notice that $p(t) \in [0, 1]^N$ for $t \geq 0$ since $\mathcal{F}^{\nu}_{\text{box}}$ is invariant to our dynamics.

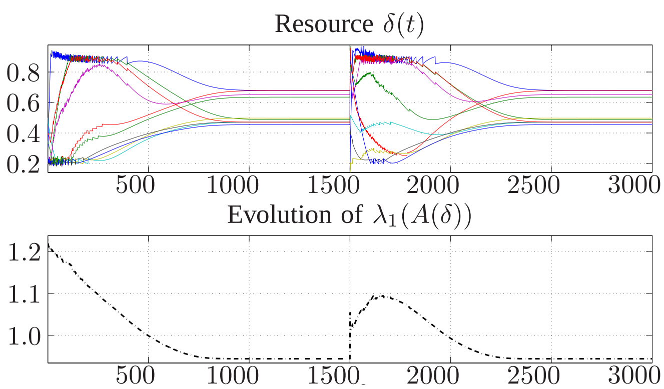

Fig. 1. Trajectories of $\delta(t)$ and $\lambda_1(A(\delta(t)))$ of Example 1 for the ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm. It is used an erroneous update on the system state at $t = 1500$.

VI. Conclusion

We have considered a class of distributed resource allocation problems. We have proposed two novel discrete-time algorithms that converge in a practical way to the solution as long as the chosen stepsize is sufficiently small. In particular, the proposed algorithms are designed to be robust to temporary errors in communication or computations of agents. Our technical approach relies on results from algebraic graph theory, second-order convex analysis as well as nonsmooth partial stability. Simulations show that the algorithms converge for a wider set of problems. Motivated by applications to virus processes, we plan to extend available proofs that can help us relax the assumptions needed.

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