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Gradient-free distributed resource allocation via simultaneous perturbation

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Abstract— **This note considers a class of decentralized convex optimization problems subject to constraints. We propose a discrete-time algorithm with constant step-size that exploits the simultaneous perturbation method to obtain information of the cost function. Under some technical conditions, we prove practical convergence in probability of the algorithm to a ball that contains the optimizer and which has a step-dependent size. The novelty of our approach is that the agents do not require a closed form expression of the cost function, nor global knowledge of total resources in the network or any specific procedure for algorithm initialization. Our proof methods employ nonsmooth Lyapunov theory, convex analysis, and stochastic difference inclusions. We illustrate the applicability of the algorithm in an electricity market scenario.**

I. Introduction

The operation of large-scale systems imposes new demands and challenges on the design of learning algorithms for optimal resource allocation. In a typical scenario, a group of agents decides how to allocate a set of limited resources to solve a common objective while satisfying operational and limited communication constraints. The challenge is in the design of algorithms that are scalable, robust against errors in communication or computation, preserve privacy, and that allow the agents in the network an autonomous decision on resource utilization. Different factors increase the complexity of such design in a network, such as problem uncertainty arising from other environmental and operational decisions that makes it difficult to the agents to have a closed form expression of their objective functions, which are often only accessible through (noisy) measurements.

A prominent application of resource allocation can be found in electricity markets, where renewable energy sources are integrated into electricity grids. In these scenarios, a main objective is to match supply with demand to compensate for power grid imbalances with distributed energy resources. Depending on what technology and source of energy is available at a given moment, the functional cost of supply and consumption may not be fully known and only numerically computable with given inputs. Costs may also depend on external prices that are determined on the global production and demand, which functional form may not be directly available. Motivated by this, here we propose a stochastic, distributed, and robust algorithm in discrete-time for resource allocation, which does not require any specific knowledge of the closed-form of agents' cost functions or any special initialization procedure.

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Literature review. The subject of model-free optimization is receiving increasing interest as it handles uncertainties in the model while providing stability guarantees. Extremum seeking control, which theoretical foundation is given in [1] for continuous-time and [2] for discrete-time, is a well-known method that has been successfully applied in real-time optimization when the analytic cost function is unknown. This is the case of [3], where the authors consider a convex optimization problem subject to linear constraints. They propose a continuous-time algorithm that combines the idea of saddle point dynamics with extremum seeking control to converge to the set of saddle points of the Lagrangian associated with the convex optimization problem in a practical way. A similar approach is followed in [4], where a primal-dual continuous-time algorithm is amended with extremum seeking control to converge to the minimizer. As an application, the authors study the dynamics of electricity markets by means of a resource allocation problem. More related to this work is the paper [5], where the authors adapt three different stochastic approximation algorithms that use a gradient based method for extremum seeking control of a dynamical system. The three papers above have in common the centralized approach to their respective optimization problems. On the other hand, the authors in [6] propose a decentralized algorithm to solve a resource allocation problem. They propose and analyze a continuous-time algorithm that combines the ideas of replicator dynamics and extremum seeking. However, the optimization problem there does not consider box constraints, the graph topology is undirected, and the continuous-time nature of the algorithm makes it difficult to implement for real-time applications.

Employing a different idea, but similar in spirit to extremum seeking control, the simultaneous perturbation (SP) gradient estimate, proposed first in [7], has been applied to optimization. The SP method is a well-known method for estimating the gradient of a cost function from noisy measurements. Its application to unconstrained optimization is known as the simultaneous perturbation stochastic approximation (SPSA) algorithm. An application of the SPSA algorithm is found in [8], [9], were the authors design a controller to drive a robot to a source without the use of the position information. The SP method has been also applied to constrained optimization; for example [10] presents a stochastic approximation algorithm based on the penalty function method to solve a optimization problem with inequality constraints. However, the algorithm in the

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papers mentioned above are based on standard assumptions found in SPSA, such as a monotonically decreasing step-size. Although this approach fits very well in many applications where direct measurements of the gradient are not available, in on-line optimization a decreasing step-size is not desirable. That is the case in source-seeking problems, where a mobile robot might get trapped in a location where the magnitude of the gradient is small or in applications where it is required to track a time-varying optimum. To overcome this drawback, a constant step-size is used. That is the case in [11], [12], where the authors propose a local voting algorithm with constant step-size and apply it to solve an approximate consensus problem for stochastic networks. However, the optimization problem there does not consider box constraints and the graph topology is undirected.

Statement of contributions. We propose and analyze a novel distributed discrete-time stochastic algorithm to solve a class of distributed resource allocation problems. Our algorithm builds on our previous work [13] and the SP method. In particular, we extend our ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS algorithm with an SP technique but, unlike previous works, we employ a constant step-size. In this way, our approach allows an interconnected group of agents to collectively minimize a global cost function subject to both equality and inequality constraints, where the closed-form expression of the local cost functions is unknown to the agents. Under some technical conditions, we show that the algorithm converges in probability to a small neighborhood of the solution as long as the chosen step-size is sufficiently small. It is shown that the proposed algorithms are convergent to a neighborhood around the equilibrium even when there are temporary errors in communication or computation. Thus, agents do not require global knowledge of total resources in the network or employ any special procedure for initialization. Our algorithm is provable correct over weight-balanced and strongly connected networks. In the proofs, we employ Lyapunov theory together with tools from convex analysis and stochastic difference inclusions.

II. PRELIMINARIES

This section presents notation and basic notions from graph, matrix, and stability theory that are used in the sequel.

A. Notation and graph-theoretic notions

In what follows, we denote by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, $\mathbb{Z}_{>0}$ the set of positive integers, $\mathbb{N}$ the set of natural numbers, and $\mathbb{R}^N_{>0}$ the positive orthant of $\mathbb{R}^N$, for some $N \in \mathbb{N}$, respectively. The spectrum of a matrix $A \in \mathbb{R}^{N\times N}$ is denoted by $\operatorname{spec}(A)$, and an eigenvalue of a symmetric matrix $A \in \mathbb{R}^{N\times N}$ is denoted by $\lambda_i(A) \in \operatorname{spec}(A)$, where $\lambda_1(A) \geq \ldots \geq \lambda_N(A)$. The singular values of A are represented by $\sigma_1(A) \geq \ldots \geq \sigma_N(A)$. In what follows, we let I_N be the identity matrix of size $N \times N$, and $\operatorname{diag}(a_1, \ldots, a_N)$ the $N \times N$ matrix with entries a_i along the diagonal. The vector $\mathbf{1}_N \in \mathbb{R}^N$ is the column vector whose elements are all equal to one. When using inequalities for vectors, these refer to componentwise inequalities. When a vector $x \in \mathbb{R}^N$ is composed by nonzero elements, we define the elementwise inverse as $x^{-1} \triangleq [x_1^{-1}, \ldots, x_n^{-1}]^\top$. We let $[l]_+ = \max\{0, l\}$, for $l \in \mathbb{R}$. The two-norm and ∞-norm of a vector are denoted by $\|.\|_2$ and $\|.\|_\infty$, respectively. A function f is $o(h)$, and we write $f(x) = o(h(x))$ as $x \to x_0$, to mean that $\lim_{x\to x_0} \frac{f(x)}{h(x)} = 0$. A function $\varphi : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. Furthermore, ϕ is *class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. The closed and open unit balls centered at the origin in $\mathbb{R}^n$ are denoted by $\mathbb{B}$ and $\mathbb{B}^\circ$, respectively.

A matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}_{\geq 0}$ is called *nonnegative* if $a_{ij} \geq 0$, for all $i, j \in \{1, \ldots, N\}$. A directed graph of order N or *digraph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$, the *vertex set*, is a set with N nodes, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, the *edge set*, is a set of ordered pair of vertices called edges. Given $B \in \mathbb{R}^{N\times N}_{\geq 0}$, its associated *weighted digraph* $\mathcal{G}(B)$ is the graph with $\mathcal{V} = \{1, \ldots, N\}$ and edge set defined by the following relationship: $(i, j) \in \mathcal{E}(B)$ if and only if $b_{ij} > 0$. The associated weight of the edge (i, j) is given by the entry b_{ij}. The digraph $\mathcal{G}(B)$ is said to be *weight-balanced* if $\sum_{j=1}^N b_{ij} = \sum_{j=1}^N b_{ji}$ for all $i \in \mathcal{V}$. Given a pair of indices $i, j \in \mathcal{V}$ of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, j is called an *out neighbor* of i if $(i, j) \in \mathcal{E}$. We let $\mathcal{N}_i^{\text{out}}(\mathcal{G})$ denote the set of out neighbors of i in $\mathcal{G}$. A digraph $\mathcal{G}(A)$ is *strongly connected* if there exists a path between any two vertices. The strongly connectedness of $\mathcal{G}(A)$ is equivalent to requiring that A is an *irreducible matrix*. The *Laplacian matrix* associated to a digraph $\mathcal{G}(A)$ is defined as $L(\mathcal{G})_{ii} = \sum_{j=1}^N a_{ij}$, and $L(\mathcal{G})_{ij} = -a_{ij}$ for $i \neq j$. Let $x \in \mathbb{R}^N$, we denote $\operatorname{Avg}(x) = \frac{1}{N}\mathbf{1}_N^\top x$.

B. Stability for stochastic difference inclusions

The notions we introduce here follow [14]. Consider a discrete-time, stochastic difference inclusion

$$x^+ \in \mathcal{H}_\alpha(x, v^+), \quad v \sim \mu, \tag{1}$$

where x^+ is the state after an instantaneous change, $\mathcal{H}_\alpha : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a set-valued map for some $n, m \in \mathbb{Z}_{>0}$ parameterized by $\alpha \in \mathbb{R}_{>0}$ and which assigns non-empty set values, and $x \in \mathbb{R}^n$ is the state. The notation v^+ and v refers to sequences of random input variables as explained next. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the set of all possible outcomes, $\mathcal{F}$ is the σ-field associated with Ω, and $\mathbb{P}$ is the probability function that assigns a probability to events in $\mathcal{F}$. In particular, we assume $\mathbf{B}(\mathbb{R}^m) \subseteq \mathcal{F}$, where $\mathbf{B}(\mathbb{R}^m)$ is the Borel field. In (1), we use v^+ and v as a place holder for a sequence of independent, identically distributed (i.i.d.) random variables $\mathbf{v} \triangleq \{\mathbf{v}_k\}_{k=0}^\infty$; that is, such that $\mathbb{P}(\mathbf{v}_k \in F) = \mathbb{P}(\{w \in \Omega \mid \mathbf{v}_k(w) \in F\})$ is well defined and independent of k for each $F \in \mathbf{B}(\mathbb{R}^m)$. We use $\mathcal{F}_k$ to denote the collection of sets $\{w \in \Omega \mid (\mathbf{v}_0(w), \ldots, \mathbf{v}_k(w)) \in F\}$, $F \in \mathbf{B}((\mathbb{R}^m)^{k+1})$, which are the sub-$\sigma$-fields of $\mathcal{F}$ that form the minimal filtration of the sequence $\mathbf{v}$. Due to the i.i.d property, each random variable has the same probability measure $\mu : \mathbf{B}(\mathbb{R}^m) \to [0, 1]$ defined as $\mu(F) \triangleq \mathbb{P}(\mathbf{v}_k \in F)$ and, for almost all $w \in \Omega$,

$$E[f(\mathbf{v}_0,\ldots,\mathbf{v}_k,\mathbf{v}_{k+1})|\mathcal{F}_k](w)$$
$$=\int_{\mathbb{R}^m} f(\mathbf{v}_0(w),\ldots,\mathbf{v}_k(w),v)\mu(dv),$$

for each $k\in\mathbb{Z}_{\geq 0}$ and each measurable $f:(\mathbb{R}^m)^{k+2}\to\mathbb{R}$.

The sequence of random variables $\mathbf{x}\triangleq\{\mathbf{x}_k\}_{k\geq 0}$, where $\mathbf{x}_k:\operatorname{dom}\mathbf{x}_k\subset\Omega\to\mathbb{R}^n$, $k\in\mathbb{Z}_{\geq 0}$ with $\mathbf{x}_0=x$ for all $w\in\Omega$ and $\operatorname{dom}\mathbf{x}_{k+1}\subset\operatorname{dom}\mathbf{x}_k$, is called a random process starting at $x\in\mathbb{R}^n$. We say that $\mathbf{x}$ is *adapted to the natural filtration* of $\mathbf{v}$ if $\mathbf{x}_{k+1}$ is $\mathcal{F}_k$ measurable for each $k\in\mathbb{Z}_{\geq 0}$, i.e., $\mathbf{x}_{k+1}^{-1}(F)\in\mathcal{F}_k$ for each $F\in\mathbf{B}(\mathbb{R}^m)$. A random process $\mathbf{x}$ starting from $x\in\mathbb{R}^n$ that is adapted to the natural filtration of $\mathbf{v}$, together with a random variable $I_{\mathbf{x}}:\Omega\to\mathbb{Z}_{\geq 0}\cup\{\infty\}$ (which denotes the number of elements in the sequence $\mathbf{x}$) is a *random solution* of (1) starting at $x\in\mathbb{R}^n$, denoted as $\mathbf{x}\in\mathcal{S}(x)$, if $\mathbf{x}_0=x$, $\mathbf{x}_{k+1}(w)\in\mathcal{H}_\alpha(\mathbf{x}_k(w),\mathbf{v}_{k+1}(w))$ for all $w\in\operatorname{dom}\mathbf{x}_{k+1}\triangleq\{w\in\Omega\mid k+1\leq I_{\mathbf{x}}\}$ and $k\in\mathbb{Z}_{\geq 0}$. We impose the following regularity condition on $\mathcal{H}$.

Assumption 1: $\mathcal{H}$ is locally bounded and $v\mapsto\operatorname{graph}(\mathcal{H}_\alpha(\cdot,v))\triangleq\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n\mid y\in\mathcal{H}_\alpha(x,v)\}$ is measurable with closed values. •

The compact set $\mathcal{A}\subset\mathbb{R}^n$ is *stable in probability* for (1) if for each $\epsilon>0$ and $\varsigma>0$ there exists $\omega>0$ such that, for each $x\in\mathcal{A}+\omega\mathbb{B}$ and $\mathbf{x}\in\mathcal{S}(x)$, $\mathbb{P}(\operatorname{graph}(\mathbf{x})\subset(\mathbb{Z}_{\geq 0}\times(\mathcal{A}+\epsilon\mathbb{B}^\circ)))\geq 1-\varsigma$.

Next, we introduce the notion of stochastic stability called recurrence, wherein solutions return to a bounded set infinitely often. Roughly speaking, an open, bounded set is said to be recurrent if almost all solutions revisit the set infinitely often. A recurrent set is not necessarily stable in probability and recurrence does not imply that solutions stay bounded, but rather states that solutions reach a compact set with probability one.

Definition 1: (Globally Recurrent Set): An open, bounded set $\mathcal{O}\subset\mathbb{R}^n$ is said to be *globally recurrent* if $E[\prod_{i\in\mathbb{Z}_{\geq 0}}\mathbb{I}_{\mathbb{R}^n\setminus\mathcal{O}}(\mathbf{x}_i)]=0$, for each $x\in\mathbb{R}^n$ and each $\mathbf{x}\in\mathcal{S}(x)$. •

Proposition 1: [14] Consider the system (1) under Assumption 1. If there exists a radially unbounded, upper semicontinuous function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$, and a continuous function $\varrho:\mathbb{R}^n\to\mathbb{R}_{>0}$ such that

$$\int_{\mathbb{R}^m}\max_{h\in\mathcal{H}_\alpha}V(h)\mu(dv)\leq V(x)-\varrho(x), \tag{2}$$

for all $x\in\mathbb{R}^n\setminus\mathcal{O}$. Then, $\mathcal{O}$ is globally recurrent for (1). •

In order to analyze the stability properties of systems of the form (1) with respect to compact sets $\mathcal{A}\subset\mathbb{R}^n$, we introduce the notion of input-to-state stability in probability. The system (1) is said to be *input-to-state stable in probability (ISSp)* relative to $\mathcal{A}$ if 1) $\mathcal{A}$ is stable in probability when $\alpha=0$ and 2) there exists $\varphi\in\mathcal{K}_\infty$ such that, for each $\alpha>0$, the open bounded set $\mathcal{A}+\varphi(\alpha)\mathbb{B}^\circ$ is globally recurrent for (1).

C. Convex analysis notions

The notions we introduce here follow [15], [16]. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a closed, proper, and convex function for some $n\in\mathbb{Z}_{>0}$. The *subgradient* of f is the set-valued map $\partial f:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ defined by the subgradient set $\partial f(x)=\{\xi\in\mathbb{R}^n\mid f(x')\geq f(x)+\xi^\top(x'-x)\}$. We refer to $df(x)$ as the *semi-derivative* function, which is the support function of the nonempty, compact, and convex set $\partial f(x)$, i.e., $df(x)(w)=\sup\{\xi^\top w\mid\xi\in\partial f(x)\}$.

We say that f satisfies the *superquadratic growth condition* if there exists $\gamma>0$ such that

$$f(y)\geq f(x)+df(x)(y-x)+\frac{\gamma}{2}\|y-x\|^2, \tag{3}$$

for $x,y\in\mathbb{R}^n$. In particular, a strongly convex function satisfies the superquadratic growth condition, and, if f is differentiable, this condition is equivalent to assuming $\rho I_n\leq\nabla^2 f(x)$ for $x\in\mathbb{R}^n$. Similarly, we say that f satisfies the *subquadratic growth condition* if there is Γ such that

$$f(y)\leq f(x)+df(x)(y-x)+\frac{\Gamma}{2}\|y-x\|^2, \tag{4}$$

for $x,y\in\mathbb{R}^n$.

III. Problem Statement, Solution Approach, and Algorithm

In this section, we introduce the optimization problem we are set out to solve, which is followed by the proposed STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm with guaranteed convergence to their corresponding optimizer under complementary sets of assumptions.

A. Problem statement and solution approach

We consider a network of N agents connected over a digraph whose goal is to minimize the sum of local payoff functions $f_i:\mathbb{R}\to\mathbb{R}_{\geq 0}$, $i\in\{1,\ldots,N\}$, where no explicit closed-form expression of the function f_i is available, under resource constraints. The BOX-COUPLED FAIRNESS optimization problem is given by

$$\begin{aligned}\min_{p\in[\underline{p},\overline{p}]^N}&\sum_{i=1}^N f_i(p_i)\\ \text{s.t.}\quad&\mathbf{1}_N^\top p=\mathbf{1}_N^\top\bar{u},\end{aligned} \tag{5}$$

where f_i is the payoff, $p=[p_1,\ldots,p_N]^\top\in\mathbb{R}^N$ is the resource allocation, $\bar{u}_i\in\mathbb{R}$ is the input assumed to be constant that represents the available quantity of resources for each agent, $\bar{u}=[\bar{u}_1,\ldots,\bar{u}_N]^\top$, and $\underline{p},\overline{p}\in\mathbb{R}^N$ are the lower and upper limits of the optimization variable, respectively. We name the last constraint in (5) as the *box constraint*. We simply refer to the problem with the box constraint omitted as the LINEARLY COUPLED FAIRNESS optimization problem. To solve both problems we state the following assumption.

Assumption 2: (Problem assumptions): We assume that the BOX-COUPLED FAIRNESS has a unique solution and there is not explicit closed-form expression of the payoff function f_i, $i\in\{1,\ldots,N\}$. The only information available are measurements of f_i at the parameter p_i. Furthermore, we assume f_i is twice continuously differentiable and bounded below with uniform bounded gradients, i.e., there exist constant $M\in\mathbb{R}_{>0}$ such that $\max_{i\in\mathcal{V}}|\frac{\partial f_i(p_i)}{\partial p_i}|\leq M$ for $p\in\mathbb{R}^N$. An upper bound of M is assumed to be known. An agent $i\in\mathcal{V}$

should be able to measure or obtain $f_j(p_j)$ for $j \in \mathcal{N}_i^{\text{out}}$. We assume that the box constraints are explicitly given. •

Under the same assumptions as for the last problem and using the exact penalty method (see, e.g., [17]), we reformulate the BOX-COUPLED FAIRNESS problem as follows:

$$\begin{aligned} &\min_p \hat{f}(p) \\ &\text{s.t.} \quad \mathbf{1}_N^\top p = \mathbf{1}_N^\top \bar{u}, \end{aligned} \tag{6}$$

where $\hat{f}(p) \triangleq \sum_{i=1}^N f_i(p_i) + J(p)$, $J(p) \triangleq \chi \sum_{i=1}^N ([\underline{p_i} - p_i]_+ + [p_i - \overline{p_i}]_+)$, and $\chi \in \mathbb{R}_{>0}$. The next lemma characterizes the optimal solution to the BOX-COUPLED FAIRNESS optimization problem.

Lemma 1: (Solution of the BOX-COUPLED FAIRNESS *problem [18]):* Let Assumption 2, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Let $\chi \in \mathbb{R}_{>0}$ be such that

$$\chi > 2 \max_{p \in \mathbb{R}^N} \|\nabla_p f(p)\|_\infty. \tag{7}$$

Then, the solution p^* to the BOX-COUPLED FAIRNESS optimization problem satisfies

$$\zeta^* \mathbf{1}_N \in \nabla_p f(p^*) + \partial J(p^*), \tag{8a}$$

$$\mathbf{1}_N^\top p^* = \mathbf{1}_N^\top \bar{u}, \tag{8b}$$

where $\zeta \in \mathbb{R}$ is the Lagrange multiplier for the equality constraint of the BOX-COUPLED FAIRNESS problem. •

Next, we propose a distributed discrete-time algorithm which successfully converges to the solutions of the BOX-COUPLED FAIRNESS problem introduced above under the corresponding assumptions. We will refer to them as the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm.

B. Proposed algorithm

In order to solve the BOX-COUPLED FAIRNESS problem dynamically, we introduce the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm shown in Algorithm 1, where

$$\Sigma = \begin{cases} w^+ = w - \alpha L(g + \psi), & (9a) \\ p^+ = p + \alpha(-L(g+\psi) + w - p + u), & (9b) \end{cases}$$

where $w \in \mathbb{R}^N$ is an internal estimator state assumed $w(0) = 0$, $\alpha \in (0,1)$ is the step-size, L is the Laplacian matrix associated to directed graph $\mathcal{G}$, $g(p,\delta,v) \triangleq [g_1(p_1,\delta,v_1),\dots,g_N(p_N,\delta,v_N)]^\top \in \mathbb{R}^N$ with $g_i : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, for $i \in \mathcal{V}$, given as:

$$g_i(p,\delta,v) = \frac{f_i(p_i + \delta_1 v_i) - f_i(p_i - \delta_2 v_i)}{\delta_1 + \delta_2} v_i^{-1}.$$

Here, $f_i : \mathbb{R} \to \mathbb{R}$ has the same meaning as in (5), $\delta = (\delta_1,\delta_2)^\top \in \mathbb{R}^2$, the random variables $\{\mathbf{v}_k\}_{k \in \mathbb{Z}_{\geq 0}}$ take values in $\{-1,1\}^N$, $\psi \in \partial J(p)$, $u \in \mathbb{R}^N$ is defined as $u = \bar{u} + \bar{\epsilon}$, where $\bar{u}_i \in \mathbb{R}$ is the input assumed to be constant that represents the available quantity of resources for each agent, $\bar{u} = (\bar{u}_1,\dots,\bar{u}_N)^\top$, and $\bar{\epsilon} \in \mathbb{R}^N$ is defined as

$$\bar{\epsilon}_i = \begin{cases} -\epsilon, & \text{if } p_i^+ = \overline{p}_i \\ \epsilon, & \text{if } p_i^+ = \underline{p}_i \\ 0, & \text{otherwise,} \end{cases}$$

Algorithm 1 One step of the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm for agent $i \in \mathcal{V}$

$\bar{\epsilon}_i = 0$
Compute Σ_i as in (9)
if $p_i^+ = \overline{p}_i$ **then**
  $\bar{\epsilon}_i = -\epsilon$
end if
if $p_i^+ = \underline{p}_i$ **then**
  $\bar{\epsilon}_i = \epsilon$
end if
Compute p_i^+ as in (9b)

where $\epsilon \in \mathbb{R}_{>0}$ is a given small constant satisfying $\epsilon \leq \alpha$.

Remark 1: Notice that the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm does not allow to have $p_i^+ = \overline{p}_i$ or $p_i^+ = \underline{p}_i$ since it perturbs Σ_i using a small quantity $\bar{\epsilon}_i$ at any time this happens. •

Since the box constraints are explicitly given, the generalized gradient of the penalty function is directly used in the algorithm.We make the following assumption on the sequence of random variables $\mathbf{v}$.

Assumption 3: (On the characteristics of the random input): The sequence of random variables $\{\mathbf{v}_k\}_{k \in \mathbb{Z}_{\geq 0}}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbf{v}_k : \Omega \to \{-1,1\}^N$, is i.i.d. with $E[\mathbf{v}_k] = 0$ for each $k \in \mathbb{Z}_{\geq 0}$. •

In what follows we use the following notation. We refer to $\mathcal{F}^u_{\leq} \triangleq \{p \in \mathbb{R}^N \mid \mathbf{1}_N^\top p \leq \mathbf{1}_N^\top \bar{u} + N\epsilon\}$, $\mathcal{F}^u_{\geq} \triangleq \{p \in \mathbb{R}^N \mid \mathbf{1}_N^\top p \geq \mathbf{1}_N^\top \bar{u} - N\epsilon\}$, $\mathcal{F}^u \triangleq \mathcal{F}^u_{\leq} \cap \mathcal{F}^u_{\geq}$, and $\mathcal{F}^\nu_{\text{box}} = \{p \in \mathbb{R}^N \mid \underline{p} - \nu \mathbf{1}_N \leq p \leq \overline{p} + \nu \mathbf{1}_N\}$ for $\nu \in \mathbb{R}_{>0}$.

Remark 2: For the easiness of presentation we neglect the presence of noise in the observations of f_i, $i \in \mathcal{V}$. However, from the analysis in the next section, practical convergence in expected value to the equilibrium point can be achieved under appropriate statistical properties on the noise. •

IV. STABILITY ANALYSIS

In this section, we show that the equilibrium point of the STOCHASTIC BOX-CONSTRAINED GRADIENT dynamics coincide with the optimal solutions of the corresponding problem under the stated assumptions when $\mathcal{G}$ is strongly connected and weight-balanced, and $g(p,\delta,v)$ is replaced by $\nabla_p f(p)$ in (9). Theorem 1 presents the stability properties of this dynamics. Proofs for all results in this paper can be found in the extended version of this paper available online in `http://fausto.dynamic.ucsd.edu/eduardo/mainResourceSP.pdf`.

Lemma 2: (Equilibria of the STOCHASTIC BOX-CONSTRAINED GRADIENT *algorithm):* Let Assumption 2, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be a weight-balanced and strongly connected graph. Let the point p^* represent the solution of the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm. Replace $g(p,\delta,v)$ by $\nabla_p f(p)$ and let $\epsilon = 0$ in (9). Then, the point p^* is the solution of the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm if and only if there exists $\eta^* \in \mathbb{R}$ such that

$$\eta^*\mathbf{1}_N \in \partial\hat{f}(p^*), \quad (10a)$$
$$\mathbf{1}_N^\top p^* = \mathbf{1}_N^\top \bar{u}. \quad (10b)$$

•

The following lemma characterizes the invariance of $\mathcal{F}_{\leq}^u$ and $\mathcal{F}_{\geq}^u$ with respect to the STOCHASTIC BOX-CONSTRAINED GRADIENT dynamics.

Lemma 3: (Invariance of the resource constraint under (9)*):* Let Assumption 2, on the payoff characteristics for the LINEARLY COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be a weight-balanced and strongly connected graph. Assume $\alpha \in (0,1)$ in (9). Then, the sets $\mathcal{F}_{\leq}^u$ and $\mathcal{F}_{\geq}^u$ are strongly positively invariant under the STOCHASTIC BOX-CONSTRAINED GRADIENT dynamics. •

Before presenting our main results, the following lemma characterizes $g(p,\delta,v)$ in terms of the gradient of the cost function f. Lemma 5 shows that the trajectories of the STOCHASTIC BOX-CONSTRAINED GRADIENT dynamics are bounded.

Lemma 4: (SP approximation to the gradient): Let Assumption 3, on the characteristics of the random input, hold. Assume that f is convex, finite, and twice differentiable and $\delta_1+\delta_2>0$. Then

$$g_i(p,\delta,v) = \frac{\partial f(p)}{\partial p_i} + b_i + c_i, \quad (11)$$

where $b_i = \sum_{j\neq i} \frac{v_j}{v_i}\frac{\partial f(p)}{\partial p_j}$, for $i \in \{1,\dots,n\}$, $c = \frac{v^{-1}}{2(\delta_1+\delta_2)} v^\top(\delta_1^2\nabla^2 f(p^1) - \delta_2^2\nabla^2 f(p^2))v$, $p^j = p + \delta_j' v$ for some $\delta_j' \in [0,1]$ and $j \in \{1,2\}$. •

Lemma 5: (Boundedness of the STOCHASTIC BOX-CONSTRAINED GRADIENT *dynamics):* Let Assumption 2, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Let $\mathcal{G}$ be weight-balanced and strongly connected. Assume that

$$\chi > \frac{1}{\min\limits_{(i,j)\in\mathcal{E}} a_{ij}}\Big(2Md_{\text{out,max}} + \|w(0)-p(0)+\bar{u}\|_\infty\Big), \quad (12)$$

and

$$\alpha < \frac{\min_{i\in\mathcal{V}}\{\overline{p}_i - \underline{p}_i\}}{2d_{\text{out,max}}(M+\chi) + \|w(0)-p(0)+\bar{u}\|_\infty}, \quad (13)$$

where $d_{\text{out,max}} = \max_{i\in\mathcal{V}}\sum_{j=1}^N a_{ij}$. Then, there exists ν such that

$$\nu \leq \max\{|\nu_1|,|\nu_2|,\nu_3\} \quad (14)$$

where $\nu_1 = \max\{\mathbf{1}_N^\top p(0), \mathbf{1}_N^\top\bar{u} + N\epsilon\} - (N-1)\min_j \underline{p}_j$, $\nu_2 = \min\{\mathbf{1}_N^\top p(0), \mathbf{1}_N^\top\bar{u} + N\epsilon\} - (N-1)\max_j \overline{p}_j$, and $\nu_3 = \alpha(2d_{\text{out,max}}\max_p\|\nabla_p f(p) + b + c\|_\infty + 2\chi d_{\text{out,max}} + \|w(0)-p(0)+u\|_\infty)$, for which the set $\mathcal{F}_{\text{box}}^\nu$ is strongly positively invariant under the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm. •

Theorem 1: (Stability of the STOCHASTIC BOX-CONSTRAINED GRADIENT *algorithm):* Let Assumption 2, on the payoff characteristics for the BOX-COUPLED FAIRNESS problem, hold. Assume $\mathcal{G}$ be a weight-balanced and strongly connected graph. Then, for any constant input $\bar{u} \in \mathbb{R}^N$ and any initial state $p(0)$, and $w(0)=0$, the solution of the system (9) converges asymptotically in probability to an open ball of radius depending on α, $\delta_1+\delta_2$, and ϵ centered at the equilibrium point (10). •

V. APPLICATION: DISTRIBUTED ECONOMIC DISPATCH WITH UNKNOWN UTILITY FUNCTIONS

We consider an electricity market consisting of suppliers and consumers, where the profit and cost functions of power generation and consumption is unknown to the consumers and suppliers. We denote by x_i the electricity consumption of consumer $i \in \{1,\dots,n\}$. Each consumer is associated with a cost function $f_i^c : \mathbb{R} \to \mathbb{R}$. We denote by z_j the electricity production by supplier $j \in \{1,\dots,m\}$. We define N as the dimension of the state space, i.e., $N = n+m$. The suppliers have an associated cost function $f_j^s : \mathbb{R} \to \mathbb{R}$. The market clearing procedure can be formulated as

$$\begin{aligned} \min_{x,z} &\sum\nolimits_{i=1}^n f_i^c(x_i) + \sum\nolimits_{i=1}^m f_i^s(z_i) \\ \text{s.t. } &\sum\nolimits_{i=1}^n x_i \leq \sum\nolimits_{i=1}^m z_i, \\ &x \in [\underline{x},\overline{x}],\ z \in [\underline{z},\overline{z}]. \end{aligned} \quad (15)$$

Notice the optimization problem above does not have the form of the BOX-COUPLED FAIRNESS optimization problem. However, we can transform the problem into

$$\begin{aligned} \min_{x,z,s} &\sum\nolimits_{i=1}^n f_i^c(x_i) + \sum\nolimits_{i=1}^m f_i^s(-z_i) \\ \text{s.t. } &\sum\nolimits_{i=1}^n x_i + \sum\nolimits_{i=1}^m z_i + \mathbf{1}_N^\top s = 0, \\ &s \geq 0,\ x \in [\underline{x},\overline{x}],\ z \in [-\overline{z},-\underline{z}]. \end{aligned} \quad (16)$$

which allows us to use the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm, as explained next.

Remark 3: A more general equality constraint can be considered for the BOX-COUPLED FAIRNESS optimization problem. For example, if we consider the equality constraint of the form $c^\top p = \mathbf{1}_N^\top\bar{u}$, where $c_i \in \mathbb{R}_{>0}$. Then, we can use a new variable $y_i = c_i p_i$ for $i \in \{1,\dots,N\}$. The BOX-COUPLED FAIRNESS optimization problem becomes

$$\begin{aligned} &\min_{y_i\in[c_i\underline{p_i},c_i\overline{p_i}]} f(\bar{y}) \\ &\text{s.t. } \mathbf{1}_N^\top y = \mathbf{1}_N^\top\bar{u}, \end{aligned} \quad (17)$$

where $\bar{y} \triangleq [c_1^{-1}y_1,\dots,c_n^{-1}y_N]^\top$. Notice that f is still convex with respect to y since it is a composition of an affine mapping.

Remark 4: When an inequality constraint '$\leq$' is considered instead of the equality constraint for the BOX-COUPLED FAIRNESS problem, we can add N slack variables $s \in \mathbb{R}_{\geq 0}^N$ to this problem to convert the inequality constraint into an equality one. In this case, the BOX-COUPLED FAIRNESS optimization problem is equivalent to

$$\begin{aligned} &\min_{p\in[\underline{p},\overline{p}],s\in\mathbb{R}_{\geq 0}^N} f(p) \\ &\text{s.t. } \mathbf{1}_N^\top p + \mathbf{1}_N^\top s = \mathbf{1}_N^\top\bar{u}. \end{aligned} \quad (18)$$

To see that the problem (18) is equivalent to the original one, first notice that if (p,s) is feasible for the problem (18), then p is feasible for the original problem, since $\mathbf{1}_N^\top s = \mathbf{1}_N^\top\bar{u} - \mathbf{1}_N^\top p \geq 0$. Conversely, if p is feasible for the original problem, then (p,s) is feasible for the problem (18), where we take $\mathbf{1}_N^\top s = \mathbf{1}_N^\top\bar{u} - \mathbf{1}_N^\top p$.

Example 1 (Distributed electricity market): We consider an electricity market consisting in 10 consumers and 5 suppliers. This example is partially taken from [4]. Our example differs from the example in [4] since we consider a graph for the topology of the electricity market. We illustrate the response of the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm for the undirected adjacency matrix A. We construct $\mathcal{G}(A)$ as a ring with $\mathcal{V} \in \{1, \ldots, 15\}$, bidirectional edges given by $a_{i,i+1} = 1/5$ for $i \in \mathcal{V}$ (assume that if $i = 15$, then $i + 1 = 1$) and additional bidirectional edges given by $a_{1,5} = a_{3,9} = 1/6$. We use $\bar{u}_i = 0$ for $i \in \mathcal{V}$, $x \in [0, 10]^n$, $z \in [-10, 0]^m$, $\chi = 10$, and $\delta_1 = \delta_2 = \alpha = 0.01$. We use the variable z in the negative orthant. Its sign means the direction of the flow, i.e., the variable is always negative indicating that it is a supplier. Figure 1 shows the behavior of the STOCHASTIC BOX-CONSTRAINED GRADIENT algorithm for a random initial condition with $x_i(0) = z_j(0) = 0$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, $w(0) = 0$. We have introduced additive Gaussian noise in the measurements of the signal mapping of zero mean and variance .05. The optimal is given by $x^* = [2.09, 1.78, 6.35, 4.46, 3.17, 2.34, 4.35, 4.08, 7.79, 2.45]$ and $z^* = [-6.63, -6.82, -7.85, -8.28, -9.27]$ (in kW). To illustrate the algorithm robustness, we introduce an erroneous update on the system state at iteration $k = 15 \times 10^3$, where we force $x(k)$ and $z(k)$ to be zero during 100 iterations. Notice that trajectories converge to the desire equilibrium no matter the erroneous updates on the system state we have introduced.

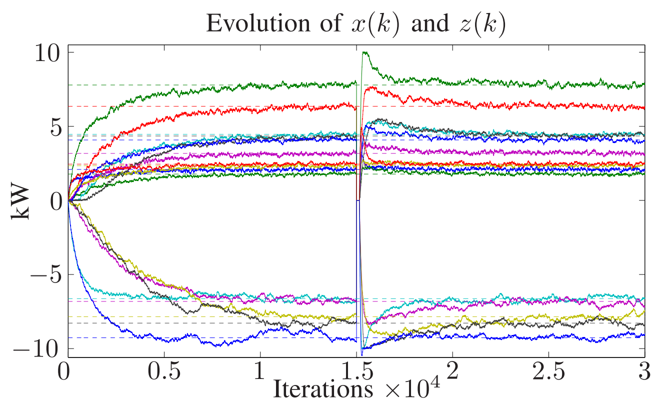

Fig. 1. Evolution of the electricity market for Example 1. Negative trajectories correspond to the suppliers $z(k)$ and positive ones to the consumers $x(k)$. Dashed lines are the optimal consumption and production x^* and z^*, respectively. In $k = 15 \times 10^3$ it is forced the states $x(k)$ and $z(k)$ to be zero during an interval of 100 iterations.

VI. CONCLUSIONS

Building on the our previous algorithm ROBUST BOX-CONSTRAINED GRADIENT FAIRNESS and the simultaneous perturbation method, we have introduced a novel stochastic algorithm that allows a group of agents to find the minimizer of an unknown cost function while satisfying inequality and equality constraints. We have proven convergence in a practical way to the solution as long as the chosen step-size is sufficiently small. In particular, the proposed algorithm are designed to be robust to temporary errors in communication or computations of agents. Our technical approach relies on results nonsmooth Lyapunov theory, convex analysis, and stochastic difference inclusions. Motivated by applications to resource allocation and optimization, we plan to extend available proofs that can help us relax the assumptions needed.

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