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Constrained source seeking for mobile robots via simultaneous perturbation stochastic approximation

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Abstract— **This paper considers a class of stochastic source seeking problems to drive a mobile robot to the maximizer of a source signal by only using measurements of the signal at the robot location. Our algorithm builds on the simultaneous perturbation stochastic approximation idea to obtain information of the signal field. We prove the practical convergence of the algorithm to a ball of size depending on the step-size that contains the location of the source. The novelty of our approach is that we consider nondifferentiable convex functions, a fixed step-size, and the environment can be restricted to any compact convex set. Our proof methods employ nonsmooth Lyapunov theory, tools from convex analysis and stochastic difference inclusions. Finally, we illustrate the applicability of the proposed algorithm in a 2D scenario for the source seeking problem.**

I. Introduction

Source seeking algorithms are used in mobile robotics for reaching the source of a radiation-like signal when position measurements are not available. Applications range from biology, in understanding bacterial foraging, to security, for rescue operations and chemical detection. In a typical scenario, the robot takes measurements of the signal emitted by the source by exploring the environment through a stochastic motion. This information is used to navigate and climb the gradient of the signal field, where the signal field might represent the spatial distribution of magnetic force, thermal signal, or chemical concentration.

Our approach is inspired by the simultaneous perturbation stochastic approximation (SPSA) method. The SPSA algorithm is a well-known method for estimating the gradient from noisy measurements of a cost function. It was originally proposed in [1] and has been successfully applied in many optimization problems like statistical modeling, parameter estimation, simulation optimization, and stochastic optimization. The original SPSA algorithm assumes a monotonically decreasing step-size, which for practical implementation in mobile robots is not an option, since it is impossible to navigate with infinite precision, which is the case when the step-size is converging to zero. Therefore, we propose a modified version of the SPSA algorithm which uses a small, but constant step-size in a constrained environment.

Literature review. There are many approaches to stochastic source seeking for mobile robots in GPS-denied environments such as the application of the extremum seeking framework to nonholonomic vehicles as in [2], or the application of the SPSA algorithm to mobile robots in [3], [4]. We follow the approach of [3], where the authors design a controller to drive a robot to the source by applying the SPSA algorithm without the use of the position information. The algorithm they propose uses standard assumptions found in SPSA in the literature such as thrice differentiability of the cost function and monotonically decreasing step-size. Those assumptions fit very well in many applications where direct measurements of the gradient are not available. However, in some applications, a decreasing step-size is not an option. An alternative is to use a small, but constant step-size, which has been successfully applied in applications such as optimization of combustion control [5], mobile robots [4], and tracking and adaptive control [6]. In [5], a variation of the SPSA algorithm is proposed which decreases the oscillation against the constraints. The proposed algorithm is applied to an automotive combustion engine problem. Although [5] uses a constant step-size, no theoretical guarantees are given for fixed step-size. In [4], a model-free algorithm is proposed based on stochastic approximation to find a source in environments with obstacles, which uses a constant step-size. A decreasing step-size is not desirable because the robot might get trapped in a location where the magnitude of the gradient is small. The convergence of the algorithm in [4] is implemented in a real world scenario. However, its convergence is not proven theoretically. In [6], an algorithm inspired by SPSA is proposed for unconstrained optimization. The algorithm uses a constant step-size to minimize a cost function for tracking problems. A drawback is that the cost function is assumed to be once differentiable and it solves an unconstrained optimization problem.

Statement of contributions. We propose a stochastic source seeking algorithm to drive a robot to an unknown source signal by only using measurements of the signal field. Our algorithm builds on the SPSA algorithms of [1] and [3]. The novelty of our approach is that we consider nondifferentiable convex functions, fixed step-size, and the environment can be any compact convex set. We prove practical convergence to a ball and whose size depends on the step-size that contains the location of the source. For the proof, we use Lyapunov theory together with tools from convex analysis, and stochastic difference inclusions. Our proof does not rely on stochastic approximation theory as is usually the case for algorithms in the literature based on SPSA.

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II. Preliminaries

This section presents notation, notions of convex analysis, and stochastic stability theory that are used in the sequel.

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A. Notation

We denote by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, $\mathbb{Z}_{>0}$ the set of positive integers, $\mathbb{R}^n_{>0}$ the positive orthant of $\mathbb{R}^n$, for some $n \in \mathbb{Z}_{>0}$, and I_n the identity matrix of size $n \times n$. When a vector $x \in \mathbb{R}^n$ is composed of nonzero elements, we define the elementwise inverse as $x^{-1} \triangleq [x_1^{-1}, \ldots, x_n^{-1}]^\top$. The two-norm of a vector is denoted by $\|.\|$. A function is f is $o(h)$, and we write $f(x) = o(h(x))$ as $x \to x_0$, to mean that $\lim_{x \to x_0} \frac{f(x)}{h(x)} = 0$. A function f is $O(h)$, and we write $f(x) = O(h(x))$ as $x \to x_0$, if there exists $\delta, M \in \mathbb{R}_{>0}$ such that $\|f(x)\| \leq M\|h(x)\|$ for $\|x - x_0\| \leq \delta$. For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S = \inf_{y \in S}|x - y|$ is the Euclidean distance to S. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* if $\limsup_{i \to +\infty} \phi(x_i) \leq \phi(x)$ whenever $\lim_{i \to +\infty} x_i = x$. Given two sets S and T, a *set-valued map*, denoted by $h : S \rightrightarrows T$, associates an element of S with a subset of T. The symbol $\mathbb{I}_S(x)$ denotes the indicator function of $\mathbb{I}_S$, i.e., $\mathbb{I}_S(x) = 1$ for $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. A set-valued map $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if, for each sequence $(x_i, y_i) \to (x, y)$ as $i \to +\infty$, in $\in \mathbb{R}^p \times \mathbb{R}^n$, and satisfying $y_i \in M(x_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, it holds that $y \in M(x)$. A mapping M is *locally bounded* if, for each bounded set $K \subset \mathbb{R}^p$, $M(K) \triangleq \cup_{x \in K} M(x)$ is bounded.

B. Convex analysis notions

The notions we introduce here follow [7], [8]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a closed, proper, and convex function. The *subgradient* of f is the set-valued map $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by the subgradient set $\partial f(x) = \{\xi \in \mathbb{R}^n \mid f(x') \geq f(x) + \xi^\top (x' - x)\}$. We refer to $df(x)$ as the *semi-derivative* function, which is the support function of the nonempty, compact, and convex set $\partial f(x)$, i.e., $df(x)(w) = \sup\{\xi^\top w \mid \xi \in \partial f(x)\}$. The first order expansion of f for any point x is given by

$$f(x + w) = f(x) + df(x)(w) + o(\|w\|). \tag{1}$$

We say that f satisfies the *superquadratic growth condition* if there exists $\rho > 0$ such that

$$f(y) \geq f(x) + df(x)(y - x) + \frac{\rho}{2}\|y - x\|^2, \tag{2}$$

for $x, y \in \mathbb{R}^n$. In particular a strongly convex function satisfies (2). When f is differentiable, satisfying (2) is equivalent to assuming that $\rho I_n \leq \nabla^2 f(x)$ for $x \in \mathbb{R}^n$.

C. Stability for stochastic difference inclusions

The notions we introduce here follow [9]. Consider a discrete-time, stochastic difference inclusion

$$x^+ \in \mathcal{H}_\alpha(x, v^+), \quad v \sim \mu, \tag{3}$$

where x^+ is the state after an instantaneous change, $\mathcal{H}_\alpha : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a set-valued map for some $n, m \in \mathbb{Z}_{>0}$ parameterized by $\alpha \in \mathbb{R}_{>0}$, which assigns non-empty set values, and $x \in \mathbb{R}^n$ is the state. The notation v^+ and v refers to sequences of random input variables as explained next. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the set of all possible outcomes, $\mathcal{F}$ is the σ-field associated with Ω, and $\mathbb{P}$ is the probability function that assigns a probability to events in $\mathcal{F}$. In particular, we assume $\mathbf{B}(\mathbb{R}^m) \subseteq \mathcal{F}$, where $\mathbf{B}(\mathbb{R}^m)$ is the Borel field. In (3), we use v^+ and v as a place holder for a sequence of independent, identically distributed (i.i.d.) random variables $\mathbf{v} \triangleq \{\mathbf{v}_k\}_{k=0}^\infty$, that is, $\mathbb{P}(\mathbf{v}_k \in F) = \mathbb{P}(\{w \in \Omega \mid \mathbf{v}_k(w) \in F\})$ is well defined and independent of k for each $F \in \mathbf{B}(\mathbb{R}^m)$. We use $\mathcal{F}_k$ to denote the collection of sets $\{w \in \Omega \mid (\mathbf{v}_0(w), \ldots, \mathbf{v}_k(w)) \in F\}$, $F \in \mathbf{B}((\mathbb{R}^m)^{k+1})$, which are the sub-$\sigma$-fields of $\mathcal{F}$ that form the minimal filtration of the sequence $\mathbf{v}$. Due to the i.i.d property, each random variable has the same probability measure $\mu : \mathbf{B}(\mathbb{R}^m) \to [0, 1]$ defined as $\mu(F) \triangleq \mathbb{P}(\mathbf{v}_k \in F)$ and, for almost all $w \in \Omega$, $E[f(\mathbf{v}_0, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1})|\mathcal{F}_k](w) = \int_{\mathbb{R}^m} f(\mathbf{v}_0(w), \ldots, \mathbf{v}_k(w), v)\mu(dv)$, for each $k \in \mathbb{Z}_{\geq 0}$ and each measurable $f : (\mathbb{R}^m)^{k+2} \to \mathbb{R}$.

The sequence of random variables $\mathbf{x} \triangleq \{\mathbf{x}_k\}_{k \geq 0}$, where $\mathbf{x}_k : \operatorname{dom} \mathbf{x}_k \subset \Omega \to \mathbb{R}^n$, $k \in \mathbb{Z}_{\geq 0}$ with $\mathbf{x}_0 = x$ for all $w \in \Omega$ and $\operatorname{dom} \mathbf{x}_{k+1} \subset \operatorname{dom} \mathbf{x}_k$, is called a random process starting at $x \in \mathbb{R}^n$. We say that $\mathbf{x}$ is *adapted to the natural filtration* of $\mathbf{v}$ if $\mathbf{x}_{k+1}$ is $\mathcal{F}_k$ measurable for each $k \in \mathbb{Z}_{\geq 0}$, i.e., $\mathbf{x}_{k+1}^{-1}(F) \in \mathcal{F}_k$ for each $F \in \mathbf{B}(\mathbb{R}^m)$. A random process $\mathbf{x}$ starting from $x \in \mathbb{R}^n$ and that is adapted to the natural filtration of $\mathbf{v}$, together with a random variable $J_{\mathbf{x}} : \Omega \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ (which denotes the number of elements in the sequence $\mathbf{x}$) is a *random solution* of (3) starting at $x \in \mathbb{R}^n$, denoted as $\mathbf{x} \in \mathcal{S}(x)$, if $\mathbf{x}_0 = x$, $\mathbf{x}_{k+1}(w) \in \mathcal{H}_\alpha(\mathbf{x}_k(w), \mathbf{v}_{k+1}(w))$ for all $w \in \operatorname{dom} \mathbf{x}_{k+1} \triangleq \{w \in \Omega \mid k + 1 \leq J_{\mathbf{x}}\}$ and $k \in \mathbb{Z}_{\geq 0}$. We impose the following regularity condition on $\mathcal{H}$.

Assumption 1: $\mathcal{H}$ is locally bounded and $v \to \operatorname{graph}(\mathcal{H}_\alpha(\cdot, v)) \triangleq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in \mathcal{H}_\alpha(x, v)\}$ is measurable with closed values.

Definition 1: (MSP-ES): We say that the equilibrium point of (3) is *mean-square practically exponentially stable* (MSP-ES) if there exists $\alpha^* \in (0, 1)$, positive real numbers β, $\lambda < \frac{1}{\alpha^*}$, γ and η, such that for all $\alpha \in (0, \alpha^*]$ and for all $k \in \mathbb{Z}_{\geq 0}$, we have $E[\|x_k\|^2] \leq \beta(1 - \alpha\lambda)^k \|x_0\|^2 + \gamma\alpha^\eta$.

Proposition 1: ([10]): Consider the system (3) under Assumption 1. If there exists an upper semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, positive constants c_1, c_2, λ, K, $\alpha^* \in (0, 1)$, and $\eta > 1$, such that for all $\alpha \in (0, \alpha^*)$

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2,$$

$$\int_{\mathbb{R}^m} \max_{h \in \mathcal{H}_\alpha} V(h)\mu(dv) \leq (1 - \alpha\lambda)V(x) + \alpha^\eta K, \tag{4}$$

then, the equilibrium point is MSP-ES for (3). •

III. Problem statement

This section describes the source seeking problem for GPS-denied environments which has been proposed in e.g. [2], [3], and [4]. We follow the approach of [3], except for the fact that we consider boundaries in the environment. Suppose that a point robot (or sufficiently small disc) moves in $\mathbb{R}^n$ and its motion is described in the world coordinate frame by

$$[\dot{p}, \dot{\theta}, \dot{\phi}]^\top = G(p(t), \theta(t), \phi(t))u(t), \tag{5}$$

where $G : \mathbb{R}^n \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{(n+n_1+n_2)\times m}$ is a function describing the robot dynamics, $p(t) \in \mathbb{R}^n$ and $\theta(t) \in \mathbb{R}^{n_1}$ are the translational and orientational positions in the world of coordinate frame, $\phi(t) \in \mathbb{R}^{n_2}$ and $u(t) \in \mathbb{R}^m$ are the internal posture and the control input, respectively.

Let $\mathcal{E}$ be the *environment* where the robot moves, which is assumed to be convex and compact. A *tower* broadcasts a signal, which is modeled by an intensity function f over $\mathbb{R}^n$. Let f denote the signal mapping $f : \mathbb{R}^n \to \mathbb{R}$, in which $f(p)$ yields the intensity at $p \in \mathbb{R}^n$, generated from a tower at p^*. The location of the tower p^* can be or not in $\mathcal{E}$. The environment $\mathcal{E}$ and even the signal mapping f are unknown to the robot. The robot aims to solve the following optimization problem

$$\min_{p\,\in\mathcal{E}} E[f(p)|p], \tag{6}$$

by only using measurements of $f(p(t))$. Furthermore, the robot does not know its own position and orientation. We are interested determining an algorithm with guaranteed practical convergence to a small ball containing the solution of (6) given by p^*. When $p^* \notin \mathcal{E}$ the robot should converge in practical way to a ball containing the closest point from $\mathcal{E}$ to p^*. First, we assume a contact sensor $l_{\mathcal{E}}(p)$ indicates whether the robot is touching the environment boundary (or any obstacle inside the environment) $\partial\mathcal{E}$ at the position p. Second, the robot is equipped with an intensity sensor, which indicates the strength of the signal from position p, i.e., $l_I(p) = f(p)$. Since the robot does not have position information in the coordinate frame, it is necessary to adapt (5) to a body fixed frame. The frame at time τ is given by

$$\begin{pmatrix} z(t) \\ \psi(t) \\ \varphi(t) \end{pmatrix} = \begin{pmatrix} R(-\theta(\tau))(p(t)-p(\tau)) \\ \theta(t)-\theta(\tau) \\ \phi(t) \end{pmatrix},$$

where t expresses a future time after τ, $(z(t), \psi(t), \varphi(t)) \in \mathbb{R}^n \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ are the new coordinates, and $R(-\theta(\tau))$ is the rotation matrix of an angle $-\theta(\tau)$.

IV. Proposed algorithm

This section assumes that there are no obstacles in $\mathcal{E}$, where $\mathcal{E}$ is described by a convex compact set. To find the tower p^*, we propose the following algorithm, which is similar in spirit to SPSA algorithm for fixed step-size:

$$p_{k+1} = \Pi_{\mathcal{E}}[p_k - \alpha g(p_k, \delta(p_k, R_k v_k), R_k, v_k)], \tag{7}$$

where $k \in \mathbb{Z}_{\geq 0}$. To simplify the notation for aid in analysis, we write the above algorithm as a discrete-time dynamical system as follows $p^+ = \Pi_{\mathcal{E}}[p - \alpha g(p, \delta(p, Rv), R, v)]$, where $p \in \mathbb{R}^n$ is the current state, $p^+ \in \mathbb{R}^n$ is the state at the next time step, $\Pi_{\mathcal{E}}$ is the projection on a convex compact set $\mathcal{E}$, and $g : \mathbb{R}^n \times \mathbb{R}^2 \times \mathrm{SO}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ is given as:

$$g(p, \delta(p, Rv), R, v) = \begin{cases} R\frac{f(p+\delta_1 Rv) - f(p-\delta_2 Rv)}{\delta_1+\delta_2} v^{-1}, & \text{if } \delta_1 + \delta_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $f : \mathbb{R}^n \to \mathbb{R}$ is the function to be minimized, $\alpha \in \mathbb{R}_{>0}$ is the step-size, $R \in \mathrm{SO}(n)$ is the uncertain time-varying rotation matrix (by definition is an orthogonal matrix), and $\delta = (\delta_1, \delta_2)$, $\delta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $i \in \{1, 2\}$, defined as

$$\delta_1(p, Rv) = \begin{cases} \bar{\delta}_1, & \text{if } p + \bar{\delta}_1 Rv \in \mathcal{E}, \\ \mathrm{dist}_{+Rv}(p, \partial\mathcal{E}), & \text{otherwise,} \end{cases}$$

$$\delta_2(p, Rv) = \begin{cases} \bar{\delta}_2, & \text{if } p - \bar{\delta}_2 Rv \in \mathcal{E} \\ \mathrm{dist}_{-Rv}(p, \partial\mathcal{E}), & \text{otherwise,} \end{cases}$$

where $\bar{\delta}_1, \bar{\delta}_2 \in \mathbb{R}_{\geq 0}$ are given constants satisfying $\bar{\delta}_1 + \bar{\delta}_2 > 0$, $\mathrm{dist}_{\pm Rv}(p, \partial\mathcal{E})$ is the distance between the point p and the set $\partial\mathcal{E}$ along the direction $\pm Rv$, and the random variable $\{\mathbf{v}_k\}_{k\in\mathbb{Z}_{\geq 0}}$ takes values in $\{-1, 1\}^n$. We assume that there is an algorithm which gives the distance from p to the position where the robot found the obstacle. This routine can be designed using information of the acceleration and the contact sensor $l_{\mathcal{E}}$ of the robot. We make the following assumption on the sequence of random variables $\mathbf{v}$.

Assumption 2: (On the characteristics of the random input): The sequence of random variables $\{\mathbf{v}_k\}_{k\in\mathbb{Z}_{\geq 0}}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbf{v}_k : \Omega \to \{-1, 1\}^n$, is i.i.d. with $E[\mathbf{v}_k] = 0$ for each $k \in \mathbb{Z}_{\geq 0}$.

Remark 1: For the easiness of presentation we neglect the presence of noise in the observations of f. However, from the analysis in the next section, practical convergence in expected value to the tower can still be achieved under appropriate statistical properties on the noise. •

V. Convergence analysis

In this section, we derive the convergence results for the algorithm in (7). In particular, we show practical convergence in probability to a ball with fixed radius depending on α and $\bar{\delta}_1 + \bar{\delta}_2$ under different assumptions. We are able to characterize the size of this ball under the assumption of strong convexity of the cost function as shown in Theorem 1. However, when we do not have enough information on the cost function, like differentiability, we prove practical convergence in probability to a ball that can be made arbitrarily small by tuning α and $\bar{\delta}_1 + \bar{\delta}_2$ as shown in Theorem 2. We begin by providing two supporting lemmas.

Lemma 1: (SPSA approximation to the gradient): Let Assumption 2, on the characteristics of the random input, hold. Assume that f convex, finite, and twice differentiable. Then, if $\delta_1 + \delta_2 > 0$ we have

$$g_i(p, \delta, R, v) = \frac{\partial f(p)}{\partial p_i} + b_i + c_i, \tag{8}$$

where $b_i = \sum_{l,j,q,j\neq l} R_{il} R_{qj} \frac{v_j}{v_l} \frac{\partial f(p)}{\partial p_q}$, for $i \in \{1, \ldots, n\}$, $c = \frac{Rv^{-1}}{2(\delta_1+\delta_2)} v^\top R^\top (\delta_1^2 \nabla^2 f(p_1) - \delta_2^2 \nabla^2 f(p_2)) Rv$, $p_j = p + \delta_j' Rv$ for some $\delta_j' \in [0, 1]$ and $j \in \{1, 2\}$. Otherwise, if $\delta_1 + \delta_2 = 0$, we have $g_i(p, \delta, R, v) = 0$ for $i \in \{1, \ldots, n\}$.

Proof: For the case when $\delta_1 + \delta_2 = 0$, by definition it follows that $g_i(p, \delta, R, v) = 0$. Otherwise, when $\delta_1 + \delta_2 > 0$, by using a second-order Taylor expansion around p, there exists $\delta_1' \in [0, 1]$ such that

$$\begin{aligned} f(p + \delta_1 Rv) = {} & f(p) + \delta_1 v^\top R^\top \nabla_p f(p) \\ & + \frac{1}{2}\delta_1^2 v^\top R^\top \nabla^2 f(p_1) Rv, \end{aligned} \tag{9}$$

where $p_1 = p + \delta_1' Rv$. Similarly, there is $\delta_2' \in [0,1]$ with $p_2 = p - \delta_2' Rv$ such that

$$f(p - \delta_2 Rv) = f(p) - \delta_2 v^\top R^\top \nabla_p f(p) + \frac{1}{2}\delta_2^2 v^\top R^\top \nabla^2 f(p_2) Rv. \tag{10}$$

Subtracting (10) from (9) and dividing the result by $\delta_1 + \delta_2$,

$$\frac{f(p+\delta_1 Rv) - f(p - \delta_2 Rv)}{\delta_1 + \delta_2} = v^\top R^\top \nabla_p f(p) + \frac{1}{2(\delta_1+\delta_2)} v^\top R^\top (\delta_1^2 \nabla^2 f(p_1) - \delta_2^2 \nabla^2 f(p_2)) Rv.$$

Multiplying last equation by Rv^{-1} we have

$$R\frac{f(p+\delta_1 Rv) - f(p - \delta_2 Rv)}{\delta_1 + \delta_2} v^{-1} = v^\top R^\top \nabla_p f(p) Rv^{-1} + \frac{Rv^{-1}}{2(\delta_1+\delta_2)} v^\top R^\top (\delta_1^2 \nabla^2 f(p_1) - \delta_2^2 \nabla^2 f(p_2)) Rv. \tag{11}$$

We analyze next the i-th component of the first term of the right-hand side (RHS) of last equation:

$$\begin{aligned}\left(v^\top R^\top \nabla_p f(p) Rv^{-1}\right)_i &= \sum_{l=1}^{n} \frac{R_{il}}{v_l} \sum_{j=1}^{n} v_j \sum_{q=1}^{n} R_{qj} \frac{\partial f(p)}{\partial p_q} \\ &= \sum_{l=1}^{n} \frac{R_{il}}{v_l} \sum_{j\neq l}^{n} v_j \sum_{q=1}^{n} R_{qj} \frac{\partial f(p)}{\partial p_q} + \sum_{l=1}^{n} R_{il} \frac{v_l}{v_l} \sum_{q=1}^{n} R_{ql} \frac{\partial f(p)}{\partial p_q} \\ &= \frac{\partial f(p)}{\partial p_i} + b_i,\end{aligned} \tag{12}$$

where we have used the fact that R is an orthogonal matrix. Replacing (12) in (11), (8) follows. ■

Lemma 2: (Optimality bounds): Assume f is convex, finite, and satisfies the superquadratic growth condition in (2). Then, for $p, p^* \in \mathbb{R}^n$ and all $\xi \in \partial f(p)$, it holds

$$(p - p^*)^\top \xi \geq \frac{\rho}{2}\|p^* - p\|^2, \tag{13}$$

and,

$$\|\xi\| \geq \frac{\rho}{2}\|p^* - p\|. \tag{14}$$

Proof: We prove first inequality (13). Given (2), it holds $f(p^*) \geq f(p) + (p^* - p)^\top \xi + \frac{\rho}{2}\|p^* - p\|^2$, for all $p, p^* \in \mathbb{R}^n$, and $\xi \in \partial f(p)$. Subtracting $f(p)$ on both sides, we have $f(p^*) - f(p) \geq (p^* - p)^\top \xi + \frac{\rho}{2}\|p^* - p\|^2$. Note that $f(p^*) - f(p) \leq 0$, then $0 \geq (p^* - p)^\top \xi + \frac{\rho}{2}\|p^* - p\|^2$. Hence, (13) follows. Next, we prove (14). Note that the RHS of (13) is bigger or equal than zero, then $|(p - p^*)^\top \xi| \geq \frac{\rho}{2}\|p^* - p\|^2$. By using the Cauchy-Schwarz inequality, it follows that $\|p - p^*\|\|\xi\| \geq \frac{\rho}{2}\|p^* - p\|^2$, which implies (14). ■

The next theorem shows algorithm convergence when f is twice differentiable.

Theorem 1: (Convergence when f is twice differentiable): Let Assumption 2, on the characteristics of the random input, hold. Assume that f is convex, finite, twice differentiable, $\rho I_n \leq \nabla^2 f(p) \leq \Gamma I_n$, and $\|\nabla_p f(p)\| \leq M$. Furthermore, assume α and $\bar{\delta}_1 + \bar{\delta}_2$ are sufficiently small. Then, for any initial state p_0, the solution p^* of the system (7) is MSP-ES with ultimate bound $\mathcal{O} = \mathcal{E} \setminus Z$, where

$$Z = \Big\{ p \in \mathcal{E} \,|\, \|p - p^*\|^2 \geq \frac{\alpha}{\rho}(M^2(n^2+2) + \frac{1}{4}(\bar{\delta}_1 + \bar{\delta}_2)^2 \Gamma^2 n^3 \Big\}. \tag{15}$$

Proof: Without loss of generality assume $\delta_1(p, Rv) + \delta_2(p, Rv) > 0$. This is the case because, at any time $k > 0$ for which $\delta_1(p, Rv) + \delta_2(p, Rv) = 0$, with probability one, the dynamics in (7) will generate a feasible direction in finite time in $\mathcal{E}$ satisfying $\delta_1(p, Rv) + \delta_2(p, Rv) > 0$. Without loss of generality assume $p^* \in \mathcal{E}$ (the projection of p^* on $\mathcal{E}$ is in $\mathcal{E}$ and unique.) By the non-expansive property of the projection operation, Algorithm (7), and the fact that $p^* \in \mathcal{E}$, we have

$$\begin{aligned}\|p^+ - p^*\|^2 &= \|\Pi_{\mathcal{E}}[p - \alpha g(p, \delta(p, Rv), R, v)] - p^*\|^2 \\ &\leq \|p - \alpha g(p, \delta(p, Rv), R, v) - p^*\|^2 \\ &= \|p - \alpha(\nabla_p f(p) + b + c) - p^*\|^2 \\ &= \|p - p^*\|^2 - 2\alpha(\nabla_p f(p) + b + c)^\top (p - p^*) \\ &\quad + \alpha^2 \|\nabla_p f(p) + b + c\|^2,\end{aligned}$$

where $b_i = \sum_{l,j,q,j\neq l} R_{il} R_{qj} \frac{v_j}{v_l} \frac{\partial f(p)}{\partial p_q}$, for $i \in \{1, \ldots, n\}$, $c = \frac{Rv^{-1}}{2(\delta_1+\delta_2)} v^\top R^\top (\delta_1^2 \nabla^2 f(p_1) - \delta_2^2 \nabla^2 f(p_2)) Rv$, $p_j = p + \delta_j' Rv$ for some $\delta_j' \in [0,1]$ and $j \in \{1, 2\}$ (see Lemma 1 to learn how to get b and c).

Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $V = \|p - p^*\|^2$, and define $\Delta V = \|p^+ - p^*\|^2 - \|p - p^*\|^2$. We have

$$\begin{aligned}\Delta V &\leq -2\alpha(\nabla_p f(p) + b + c)^\top (p - p^*) \\ &\quad + \alpha^2 \|\nabla_p f(p) + b + c\|^2.\end{aligned}$$

By using (13), we have that $-(p - p^*)^\top \nabla_p f(p) \leq -\frac{\rho}{2}\|p - p^*\|^2$. It follows that

$$\begin{aligned}\Delta V &\leq -\alpha\rho\|p - p^*\|^2 - 2\alpha(b + c)^\top (p - p^*) \\ &\quad + \alpha^2 \|\nabla_p f(p) + b + c\|^2.\end{aligned}$$

By taking expectation operator $E[V(p^+)|\mathcal{F}_k]$, since $\mathbf{v}_k$ is i.i.d with $E[\mathbf{v}_k] = 0$ for each $k \in \mathbb{Z}_{\geq 0}$, and by noticing that $E[\mathbf{v}_k^{-1}] = E[\mathbf{v}_k]$, it follows that $E[b] = 0$. Next, we show that $E[c_i] = 0$ for $i \in \{1, \ldots, n\}$. We rewrite $c = m(v^\top H v) Rv$, where $m = \frac{1}{2(\delta_1+\delta_2)}$, $H \triangleq R^\top(\delta_1^2 \nabla^2 f(p_1) - \delta_2^2 \nabla^2 f(p_2)) R$, $H = (h_{ij})$, and we use the fact that $v = v^{-1}$. Then,

$$\begin{aligned}E[c_i] &= mE[\sum_{l=1}^{n} R_{il} v_l \sum_{k=1}^{n} v_k \sum_{j=1}^{n} h_{kj} v_j] \\ &= m(q_i + z_i),\end{aligned}$$

where $q_i = E[R_{ii} v_i \sum_{k=1}^{n} v_k \sum_{j=1}^{n} h_{kj} v_j]$ and $z_i = E[\sum_{l\neq i} R_{il} v_l \sum_{k=1}^{n} v_k \sum_{j=1}^{n} h_{kj} v_j]$. Expanding q_i,

$$\begin{aligned}q_i &= E[R_{ii} v_i (v_i \sum_{j=1}^{n} h_{ij} v_j + \sum_{k\neq i} v_k \sum_{j=1}^{n} h_{kj} v_j)] \\ &= R_{ii} E[h_{ii} v_i^3 + v_i^2 \sum_{j\neq i} h_{ij} v_j + v_i \sum_{k\neq i} v_k^2 h_{kk} \\ &\quad + v_i \sum_{k\neq i} v_k \sum_{j\neq k} h_{kj} v_j] \\ &= 0,\end{aligned}$$

where we have used the assumption that $\mathbf{v}_k$ is i.i.d with $E[\mathbf{v}_k]=0$ and the fact that $v_i^3=v_i$ for $i\in\{1,\dots,n\}$. Analogous to last procedure, we expand z_i,

$$\begin{aligned}z_i &= E[\sum_{l\neq i} R_{il} v_l^2 \sum_{j=1}^{n} h_{lj} v_j + \sum_{l\neq i} R_{il} v_l \sum_{k\neq l} v_k \sum_{j=1}^{n} h_{kj} v_j] \\ &= E[\sum_{l\neq i} R_{il} v_l^3 h_{ll} + \sum_{l\neq i} R_{il} v_l^2 \sum_{j\neq l} h_{lj} v_j \\ &\quad + \sum_{l\neq i} R_{il} v_l \sum_{k\neq l} v_k^2 h_{kk} + \sum_{l\neq i} R_{il} v_l \sum_{k\neq l} v_k \sum_{j\neq k} h_{kj} v_j] \\ &= 0.\end{aligned}$$

Thus $E[c]=0$. Therefore,

$$\begin{aligned}E[\Delta V|\mathcal{F}_k] &\leq -\alpha\rho\|p-p^*\|^2 \\ &\quad+\alpha^2(\|\nabla_p f(p)\|^2 + E[\|b\|^2+\|c\|^2|\mathcal{F}_k]).\end{aligned} \tag{16}$$

Note that $E[\|c\|^2]\leq \frac{1}{4}\Gamma^2 n^3(\delta_1+\delta_2)^2$ and from (12) we have

$$\begin{aligned}&E[\|b\|^2|\mathcal{F}_k] = E[\|v^\top R^\top \nabla_p f(p) R v^{-1} - \nabla_p f(p)\|^2|\mathcal{F}_k] \\ &\leq E[\|v^\top R^\top\|^2 \|\nabla_p f(p)\|^2 \|Rv^{-1}\|^2 + \|\nabla_p f(p)\|^2 |\mathcal{F}_k] \\ &\leq M^2(n^2+1),\end{aligned}$$

where we have used $\|\nabla_p f\|\leq M$. Using above upper bounds and replacing them in (16), it follows that

$$\begin{aligned}E[\Delta V|\mathcal{F}_k] &\leq -\alpha\rho V(p) \\ &\quad+\frac{\alpha^2}{4}(\delta_1+\delta_2)^2\Gamma^2 n^3 + \alpha^2 M^2(n^2+2).\end{aligned}$$

It follows $E[\Delta V|\mathcal{F}_k]\leq -\alpha\rho V(p)+\alpha^2 J$, where $J=\frac{1}{4}(\delta_1+\delta_2)^2\Gamma^2 n^3 + M^2(n^2+2)$. Reorganizing these terms, we have

$$E[V(p^+)|\mathcal{F}_k] \leq (1-\alpha\rho)V(p)+\alpha^2 J.$$

Therefore, by Theorem 1 the equilibrium point is MSE-ES. Notice that the $\max$ inside the integral in (4) simplifies to a point because we do not have a differential inclusion.

The set $\mathcal{O}$ given in (15) follows by noticing that $E[\Delta V|\mathcal{F}_k]\leq 0$ if $\|p-p^*\|^2\geq \mathcal{O}$ and by noticing that $\delta_1+\delta_2\leq\bar{\delta}_1+\bar{\delta}_2$. ■

Remark 2: In the last theorem, knowledge of Γ and M are not required to prove convergence. Given that $\mathcal{E}$ is assumed to be compact, existence of Γ is guaranteed. Furthermore, since f is assumed locally Lipschitz, then there always exist a finite M such that $\|\nabla_p f(p)\|\leq M$. However, we use those values to characterize the size of the ball where the trajectories converge to in expectation. •

If f is nondifferentiable, we are not able to characterize the size of the ball as in Theorem 1. However, the next result shows practical convergence in probability to p^* and that this ball can be made arbitrarily small by reducing α and $\bar{\delta}_1+\bar{\delta}_2$ without the assumption on the superquadratic growth condition on f.

Theorem 2: (Convergence when f is nonsmooth): Let Assumption 2, on the characteristics of the random input, hold. Assume that α and $\bar{\delta}_1+\bar{\delta}_2$ are sufficiently small. Moreover, assume that f is convex, and finite with a unique minimizer p^*. Then, for any initial state p_0, the solution p^* of the system (7) is MSP-ES.

Proof: The proof follows along the lines of the proof of Theorem 1, except that we can not resort to the differentiability properties of f and we do not use the assumption on its superquadratic growth condition.

Since f is assumed to be convex and locally Lipschitz, then the set-valued map ∂f is locally bounded, upper semicontinuous, and takes non-empty, compact, and convex values [11]. Using the last fact, the $\sup$ in (1) can be replaced by a $\max$, and then

$$f(p+\delta_1 Rv) = f(p) + \delta_1 v^\top R^\top \bar{\xi} + o(\delta_1\|Rv\|), \tag{17}$$

where $\bar{\xi}=\mathrm{argmax}_{\xi\in\partial f_s(p)}\{\xi^\top Rv\}$. Similarly,

$$f(p-\delta_2 Rv) = f(p) - \delta_2 v^\top R^\top \underline{\xi} + o(\delta_2\|Rv\|), \tag{18}$$

where $\underline{\xi}=\mathrm{argmin}_{\xi\in\partial f(p)}\{\xi^\top Rv\}$. Subtracting (18) from (17) and dividing the result by $\delta_1+\delta_2$, we have

$$\begin{aligned}&\frac{f(p+\delta_1 Rv)-f(p-\delta_2 Rv)}{\delta_1+\delta_2} \\ &= \frac{1}{\delta_1+\delta_2}\Big(v^\top R^\top(\delta_1\bar{\xi}+\delta_2\underline{\xi}) + o(\delta_1\|v\|) - o(\delta_2\|v\|)\Big),\end{aligned}$$

where we have used the assumption that R is an orthogonal matrix, then $o(\delta_i\|Rv\|)=o(\delta_i\|v\|)$ for $i\in\{1,2\}$. Multiplying the last equation by Rv^{-1}, we have

$$\begin{aligned}&\frac{f(p+\delta_1 Rv)-f(p-\delta_2 Rv)}{\delta_1+\delta_2}Rv^{-1} \\ &= v^\top R^\top(\delta_1\bar{\xi}+\delta_2\underline{\xi})\frac{Rv^{-1}}{\delta_1+\delta_2} \\ &\quad+\frac{Rv^{-1}}{\delta_1+\delta_2}(o(\delta_1\|v\|)-o(\delta_2\|v\|)).\end{aligned} \tag{19}$$

We analyze the i-th component of the first term of the RHS of the last equation to obtain

$$\begin{aligned}&\Big(v^\top R^\top(\delta_1\bar{\xi}+\delta_2\underline{\xi})\frac{Rv^{-1}}{\delta_1+\delta_2}\Big)_i \\ &= \sum_{l=1}^{n}\frac{R_{il}}{v_l}\sum_{j=1}^{n} v_j \sum_{q=1}^{n} R_{qj}\frac{\delta_1\bar{\xi}_q+\delta_2\underline{\xi}_q}{\delta_1+\delta_2} \\ &= \frac{\delta_1\bar{\xi}_i+\delta_2\underline{\xi}_i}{\delta_1+\delta_2}+b_i,\end{aligned} \tag{20}$$

where $b_i=\sum_{l,j,q,j\neq l} R_{il}R_{qj}\frac{v_j}{v_l}\frac{\delta_1\bar{\xi}_q+\delta_2\underline{\xi}_q}{\delta_1+\delta_2}$. Replacing (20) in (19), it follows that

$$\frac{f(p+\delta_1 Rv)-f(p-\delta_2 Rv)}{\delta_1+\delta_2}Rv^{-1} = \frac{\delta_1\bar{\xi}+\delta_2\underline{\xi}}{\delta_1+\delta_2}+b+c,$$

where $c=\frac{Rv^{-1}}{\delta_1+\delta_2}(o(\delta_1\|v\|)-o(\delta_2\|v\|))$. It follows that

$$g(p,\delta(p,Rv),R,v) = \frac{\delta_1\bar{\xi}+\delta_2\underline{\xi}}{\delta_1+\delta_2}+b+c.$$

Without loss of generality assume $\delta_1(p,Rv)+\delta_2(p,Rv)>0$. This is the case because, at any time $k>0$ for which $\delta_1(p,Rv)+\delta_2(p,Rv)=0$, with probability one, the dynamics in (7) will generate a feasible direction in finite time in $\mathcal{E}$ satisfying $\delta_1(p,Rv)+\delta_2(p,Rv)>0$.

Further, without loss of generality assume $p^* \in \mathcal{E}$.

By the non-expansive property of the projection operation, the dynamics in (7), and the fact that $p^* \in \mathcal{E}$, we have

$$\begin{aligned}&\|p^+ - p^*\|^2 = \|\Pi_{\mathcal{E}}[p - \alpha g(p, \delta(p, Rv), R, v)] - p^*\|^2 \\ &\leq \|p - \alpha g(p, \delta(p, Rv), R, v) - p^*\|^2 \\ &\leq \|p - \alpha\Big(\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c\Big) - p^*\|^2.\end{aligned}$$

It follows

$$\begin{aligned}\|p^+ - p^*\|^2 &\leq \|p - p^*\|^2 \\ &- 2\alpha(\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c)^\top (p - p^*) \\ &+ \alpha^2 \|\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c\|^2.\end{aligned}$$

Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $V = \|p - p^*\|^2$, and define $\Delta V = \|p^+ - p^*\|^2 - \|p - p^*\|^2$. Then we have

$$\begin{aligned}\Delta V \leq -2\alpha\Big(\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c\Big)^\top (p - p^*) \\ + \alpha^2 \|\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c\|^2.\end{aligned}$$

Let $f_s : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying the superquadratic growth condition for some $\rho \in \mathbb{R}_{>0}$ such that $f_s(p^*) = f(p^*)$, $\xi_s^\top (p - p^*) \leq \xi^\top (p - p^*)$, $\xi \in \partial f(p)$, and $\xi_s \in \partial f_s(p)$ for all $p \in \mathcal{E}$. Notice that f_s always can be found since p^* is assumed unique and $\mathcal{E}$ is a compact set. Using the last fact, there exists $\rho > 0$ such that $-\xi^\top (p - p^*) \leq -\frac{\rho}{2}\|p - p^*\|^2$. It follows that

$$\begin{aligned}\Delta V &\leq -\alpha\rho \frac{\delta_1 + \delta_2}{\delta_1 + \delta_2}\|p - p^*\|^2 \\ &- 2\alpha(b + c)^\top (p - p^*) + \alpha^2 \|\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c\|^2.\end{aligned}$$

By noticing that $E[b] = E[c] = 0$, it follows that

$$E[\Delta V | \mathcal{F}_k] \leq -\alpha\rho\|p - p^*\|^2 + \alpha^2 \|\frac{\delta_1 \bar{\xi} + \delta_2 \underline{\xi}}{\delta_1 + \delta_2} + b + c\|^2 | \mathcal{F}_k].$$

From here, the proof follows similar steps as the proof of Theorem 1, where we use ξ instead of $\nabla_p f$, and consider $O(1)$ terms instead of the upper bound of the Hessian. ■

VI. Simulations

Next we show an example that illustrates the response of our proposed Algorithm (7) to solve a particular source seeking problem. Figure 1 illustrates the evolution of the mobile robot to a source $f = (p_1 - .9)^2 + |p_1 - .9| + (p_2 - 1)^2 + |p_2 - 1|$ with a box constraint $p \in [0, 1]^2$. Notice that the function f is nondifferentiable and strongly convex, then it satisfies the conditions on Theorem 2. The tower is located at $p^* = [.9, 1]^\top$. This simulation uses $\alpha = \bar{\delta}_1 = \bar{\delta}_2 = .02$ and $R_k = I_n$ for all $k \geq 0$. We have introduced additive gaussian noise in the measurements of the intensity signal of zero mean and variance .0001. The robot starts at $p_0 = [.6, .1]^\top$. The robot converges to a ball containing the optimizer p^*, which in turn can be made arbitrarily small by decreasing the parameters α, $\bar{\delta}_1$, and $\bar{\delta}_2$.

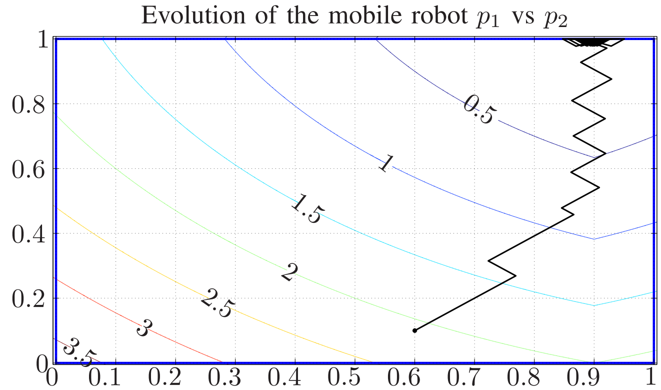

Fig. 1. Evolution of the mobile robot for $f = (p_1 - .9)^2 + |p_1 - .9| + (p_2 - 1)^2 + |p_2 - 1|$ with a box constraint $p \in [0, 1]^2$. The level sets of f are shown in colors and the trajectory of the robot is shown in black.

VII. Conclusions

Building on the simultaneous perturbation stochastic approximation method, we have introduced a novel algorithm that allows a mobile robot to find the maximizer of an emitting signal. We are able to prove convergence to a ball around the optimizer of the emitting signal, even for non-differentiable signal case and restricting the motion of the robot to a convex set. Current work is being devoted to extend the available proofs to scenarios that include obstacles in the environment as well as the relaxation of the various assumptions of the algorithms.

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