Systematic Circuit Analysis (T&R Chap 3)

Node-voltage analysis

Using the voltages of the each node relative to a ground node, write down a set of consistent linear equations for these voltages

Solve this set of equations using, say, Cramer's Rule

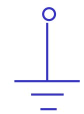
Mesh current analysis

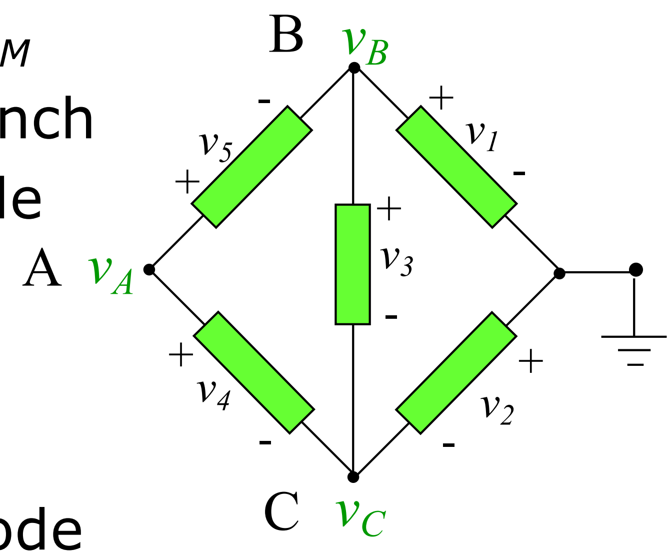
Using the loop currents in the circuit, write down a set of consistent linear equations for these variables. Solve.

This introduces us to procedures for *systematically* describing circuit variables and solving for them

Nodal Analysis

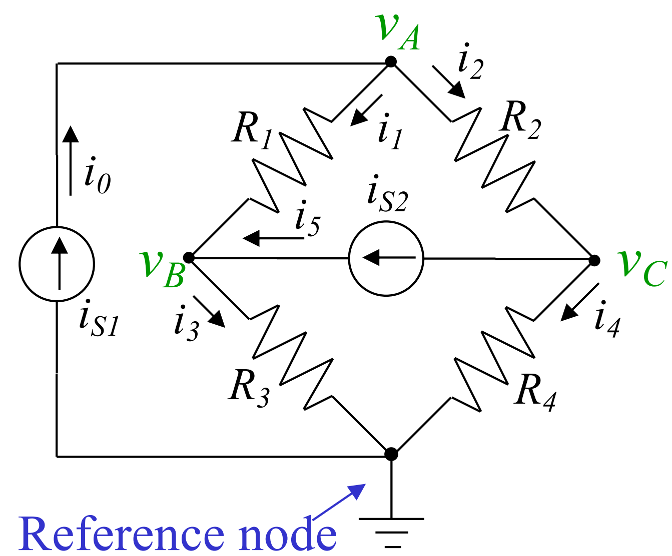
Node voltages

- Pick one node as the ground node 
- Label all other nodes and assign voltages v_A, v_B, …, v_N and currents with each branch i_1, …, i_M
- Recognize that the voltage across a branch is the difference between the end node voltages
  - Thus $v_3=v_B-v_C$ with the direction as indicated



- Write down the KCL relations at each node
- Write down the branch *i-v* relations to express branch currents in terms of node voltages
- Accommodate current sources
- Obtain a set of linear equations for the node voltages

Nodal Analysis – Ex 3-1 (T&R, 5th ed, p.72)



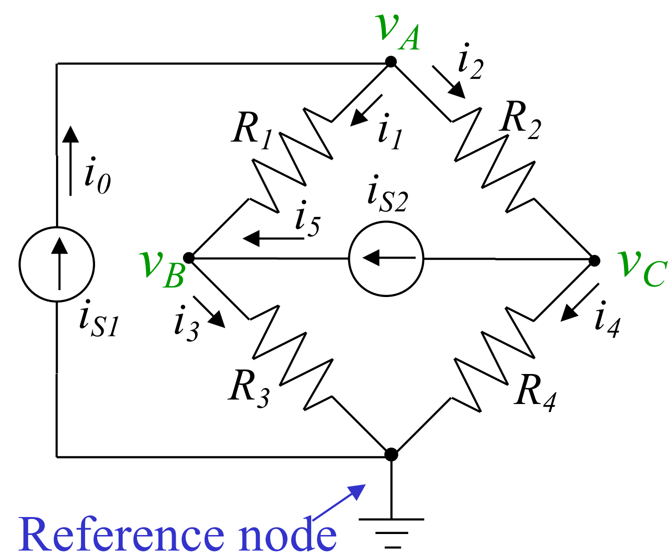
Apply KCL

Write the element/branch eqns

Substitute to get node voltage equations

Solve for v_A, v_B, v_C then i_0, i_1, i_2, i_3, i_4, i_5

Nodal Analysis – Ex 3-1 (T&R, 5th ed, p.72)



Apply KCL

Node A: $i_0 - i_1 - i_2 = 0$

Node B: $i_1 - i_3 + i_5 = 0$

Node C: $i_2 - i_4 - i_5 = 0$

Write the element/branch eqns

$i_0 = i_{S1}$	$i_3 = G_3 v_B$
$i_1 = G_1(v_A - v_B)$	$i_4 = G_4 v_C$
$i_2 = G_2(v_A - v_C)$	$i_5 = i_{S2}$

Substitute to get node voltage equations

Node A: $(G_1+G_2)v_A - G_1 v_B - G_2 v_C = i_{S1}$

Node B: $-G_1 v_A + (G_1+G_3)v_B = i_{s2}$

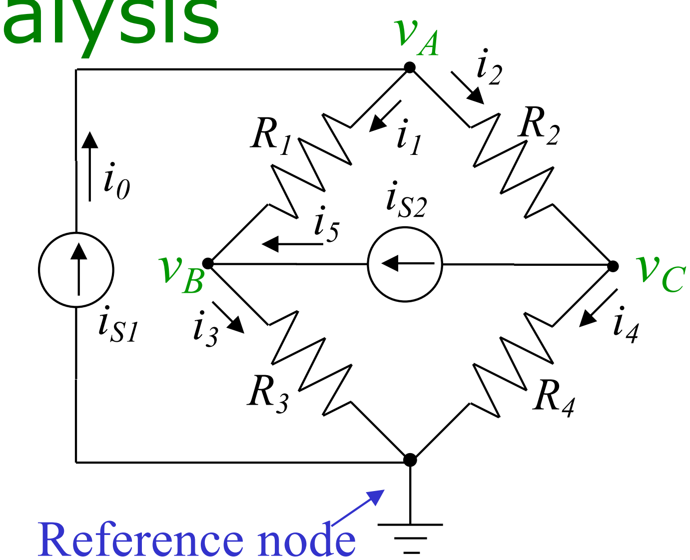
Node C: $-G_2 v_A + (G_2+G_4)v_C = -i_{S2}$

$$\begin{pmatrix} G_1+G_2 & -G_1 & -G_2 \\ -G_1 & G_1+G_3 & 0 \\ -G_2 & 0 & G_2+G_4 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{S1} \\ i_{S2} \\ -i_{S2} \end{pmatrix}$$

Solve for v_A, v_B, v_C then i_0, i_1, i_2, i_3, i_4, i_5

Systematic Nodal Analysis

$$\begin{pmatrix} G_1+G_2 & -G_1 & -G_2 \\ -G_1 & G_1+G_3 & 0 \\ -G_2 & 0 & G_2+G_4 \end{pmatrix}\begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{S1} \\ i_{S2} \\ -i_{S2} \end{pmatrix}$$



Writing node equations by inspection

Note that the matrix equation looks just like $\underline{G}\underline{v}=\underline{i}$ for matrix $\underline{G}$ and vector $\underline{v}$ and $\underline{i}$

$\underline{G}$ is symmetric (and non-negative definite)

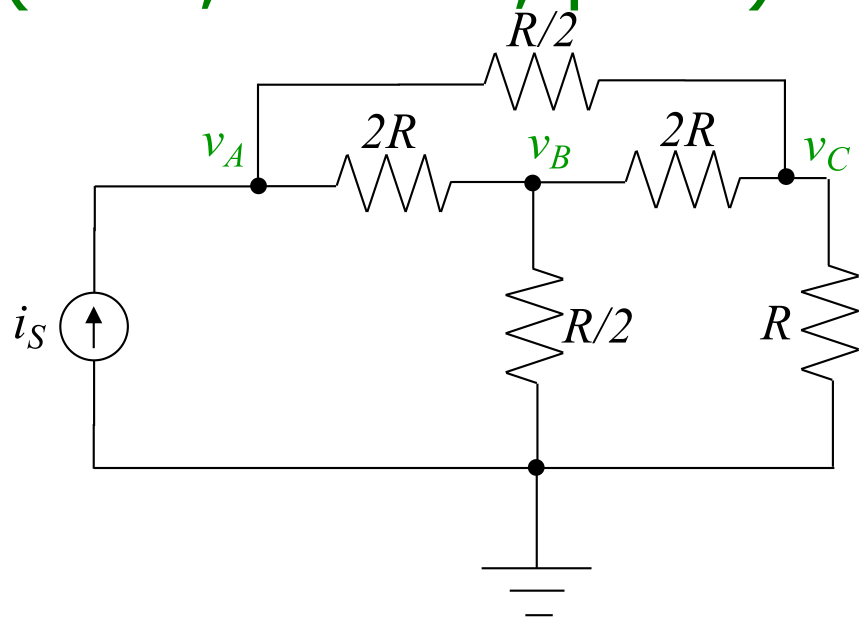
Diagonal *(i,i)* elements: sum of all conductances connected to node *i*

Off-diagonal *(i,j)* elements: -conductance between nodes *i* and *j*

Right-hand side: current sources entering node *i*

There is no equation for the ground node – the column sums give the conductance to ground

Nodal Analysis Ex. 3-2 (T&R, 5th ed, p.74)



Node A:

Conductances

Source currents entering

Node B:

Conductances

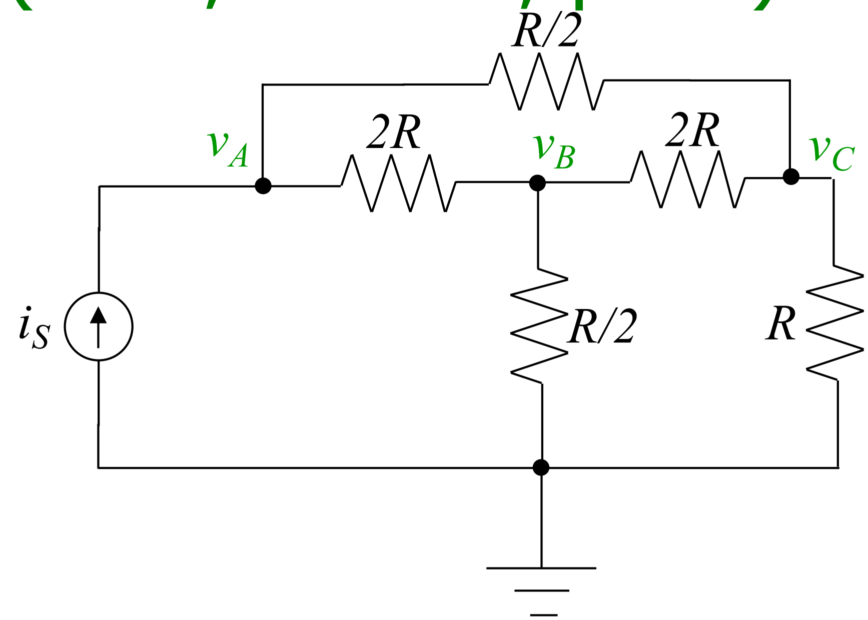
Source currents entering

Node C:

Conductances

Source currents entering

Nodal Analysis Ex. 3-2 (T&R, 5th ed, p.74)



Node A:

Conductances

$G/2_B+2G_C=2.5G$

Source currents entering= i_S

Node B:

Conductances

$G/2_A+G/2_C+2G_{ground}=3G$ Source currents entering= *0*

Node C:

Conductances

$2G_A+G/2_B+G_{ground}=3.5G$ Source currents entering= 0

$$\begin{pmatrix} 2.5G & -0.5G & -2G \\ -0.5G & 3G & -0.5G \\ -2G & -0.5G & 3.5G \end{pmatrix}\begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_S \\ 0 \\ 0 \end{pmatrix}$$

Nodal Analysis – some points to watch

1. The formulation given is based on KCL with the sum of currents ***leaving*** the node

   $0=i_{total}=G_{AtoB}(v_A-v_B)\ +G_{AtoC}(v_A-v_C)+\ldots+\ G_{AtoGround}v_A+i_{leavingA}$

   This yields

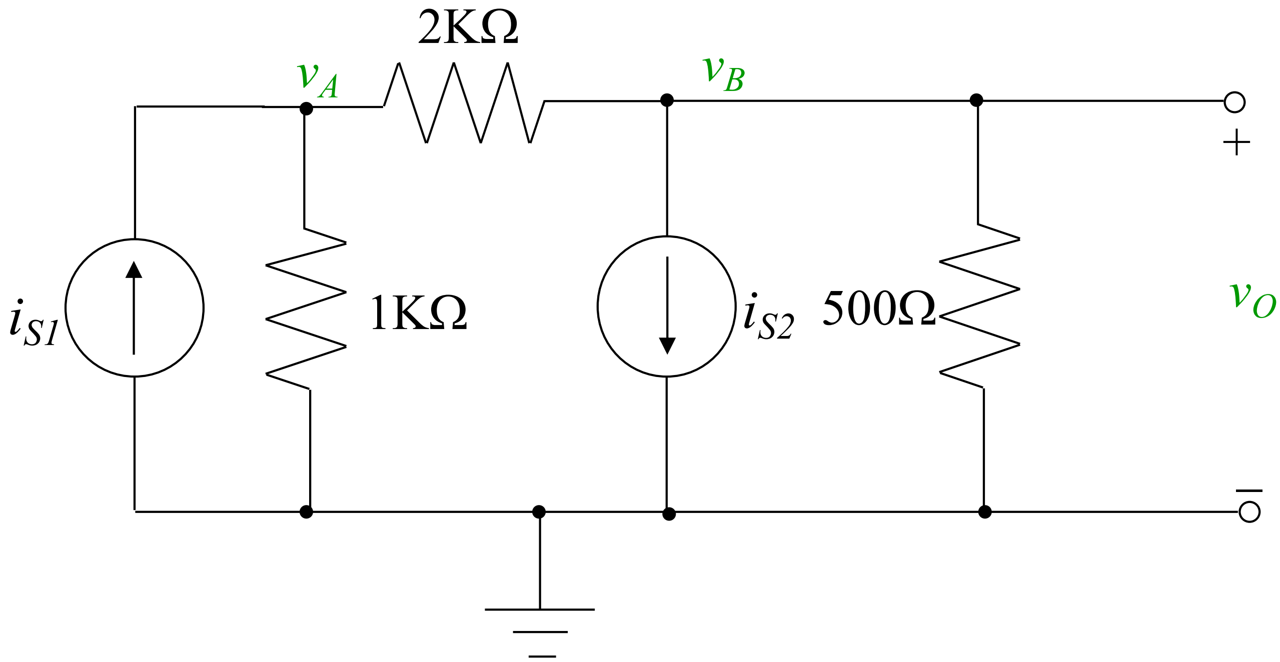
   $0=(G_{AtoB}+\ldots+G_{AtoGround})v_A-G_{AtoB}v_B-G_{AtoC}v_C\ldots-i_{enteringA}$

   $(G_{AtoB}+\ldots+G_{AtoGround})v_A-G_{AtoB}v_B-G_{AtoC}v_C\ldots=i_{enteringA}$

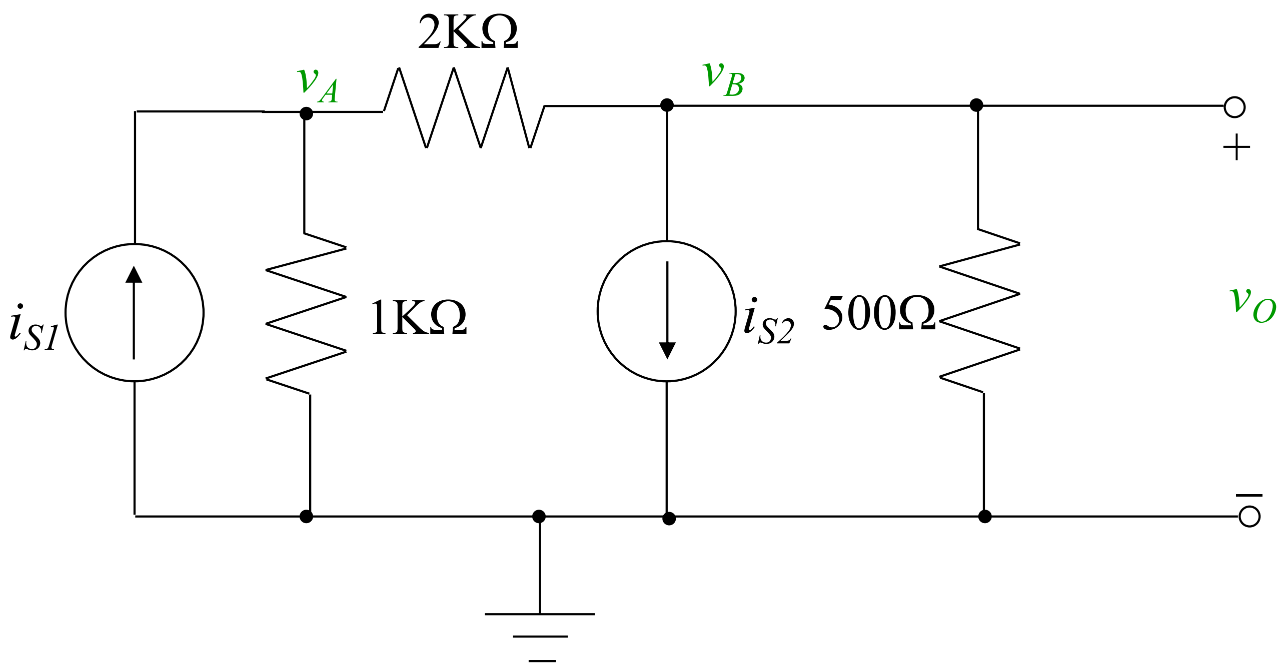
2. If in doubt about the sign of the current source, go back to this basic KCL formulation
3. This formulation works for independent current sources

   For dependent current sources (introduced later) use your wits

Nodal Analysis Ex. 3-2 (T&R, 5th ed, p. 72)



Nodal Analysis Ex. 3-2 (T&R, 5th ed, p. 72)



$$\begin{pmatrix} 1.5\times10^{-3} & -0.5\times10^{-3} \\ -0.5\times10^{-3} & 2.5\times10^{-3} \end{pmatrix}\begin{pmatrix} v_A \\ v_B \end{pmatrix} = \begin{pmatrix} i_{S1} \\ -i_{S2} \end{pmatrix}$$

Solve this using standard linear equation solvers

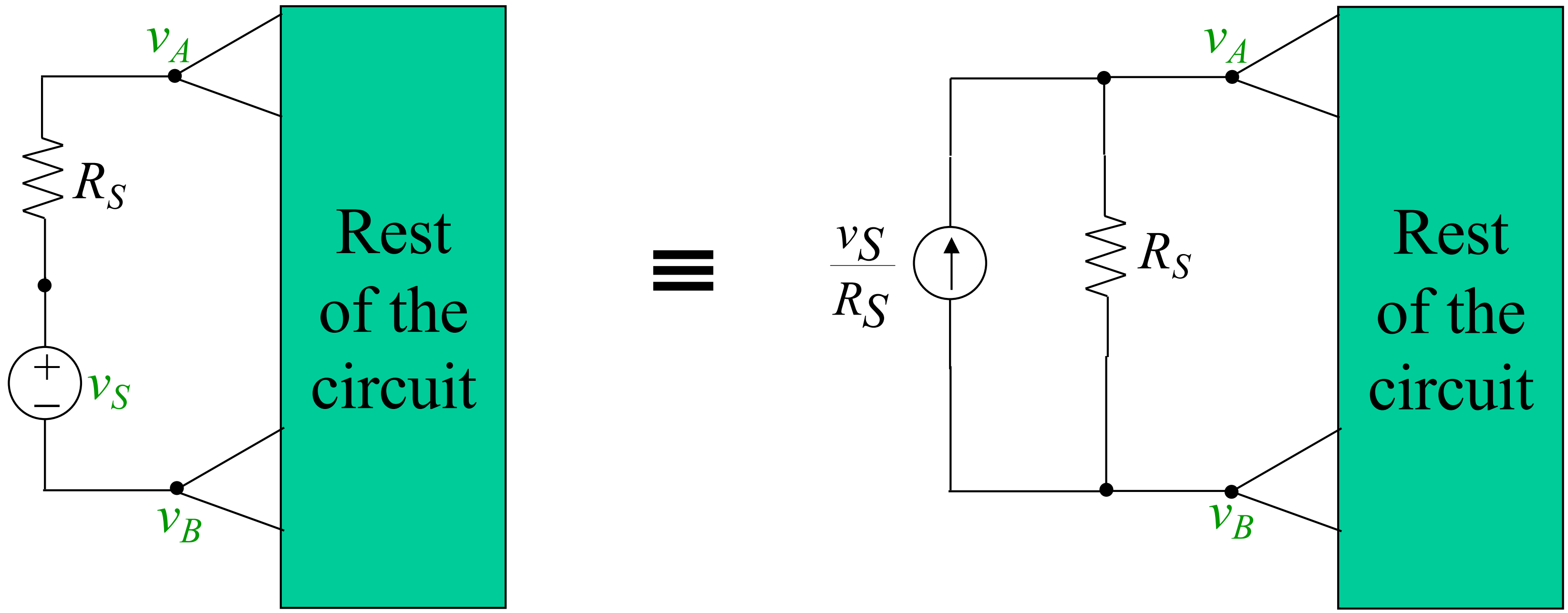
- Cramer's rule
- Gaussian elimination
- Matlab

Nodal Analysis with Voltage Sources

Current through voltage source is not computable from voltage across it. We need some tricks!

They actually help us simplify things

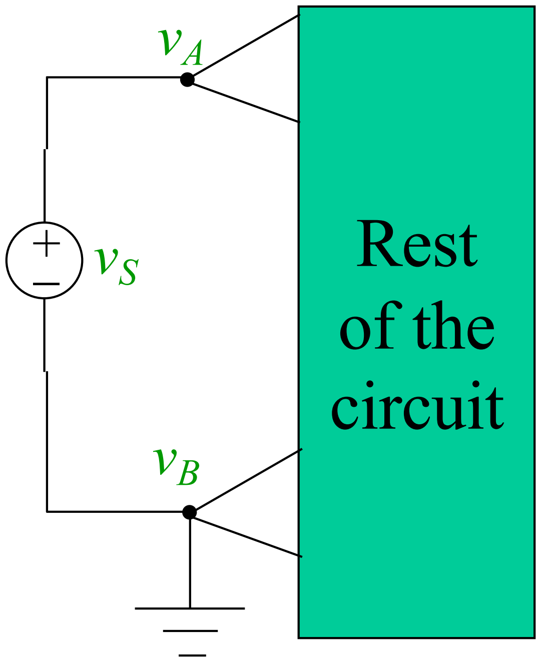
Method 1 – source transformation



Then use standard nodal analysis – one less node!

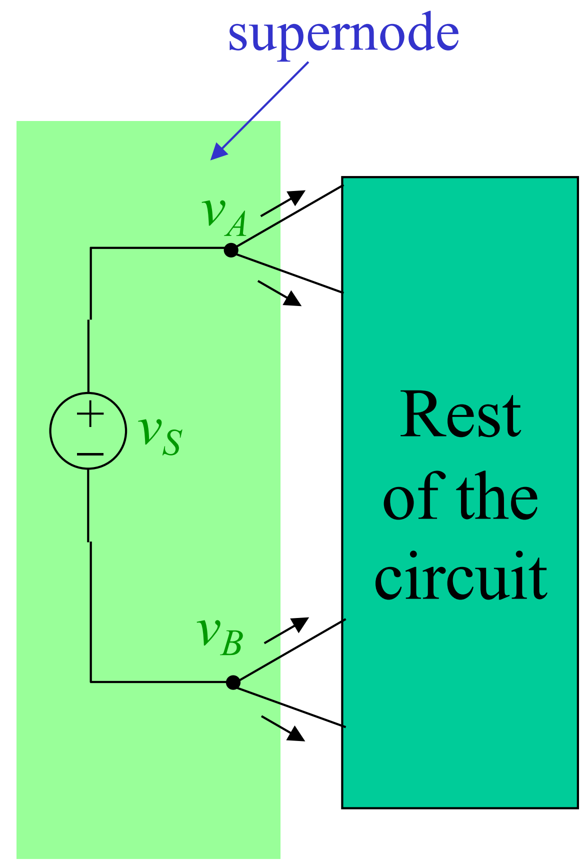
Nodal Analysis with Voltage Sources 2

Method 2 – grounding one node



This removes the v_B variable
(plus we know $v_A = v_S$) – simpler analysis
But can be done once per circuit

Nodal Analysis with Voltage Sources 3

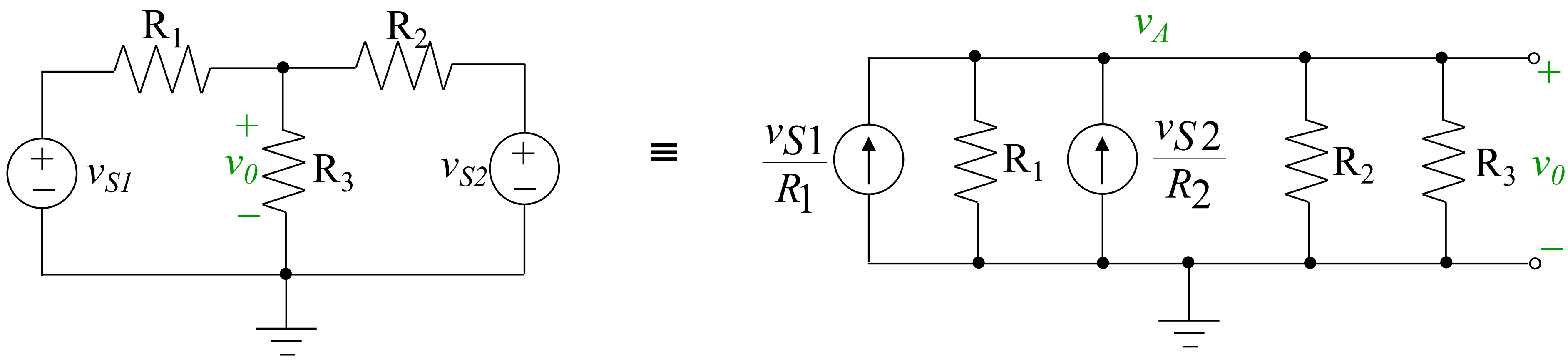


Method 3

- Create a *supernode*
  - Act as if A and B were one node
  - KCL still works for this node
    - Sum of currents entering supernode box is 0
- Write KCL at all N-3 other nodes
  - (N-2 nodes less Ground node)
  - using v_A and v_B as usual
- +Write one supernode KCL
- +Add the constraint $v_A - v_B = v_S$

These three methods allow us to deal with all cases

Nodal Analysis Ex. 3-4 (T&R, 5th ed, p. 76)

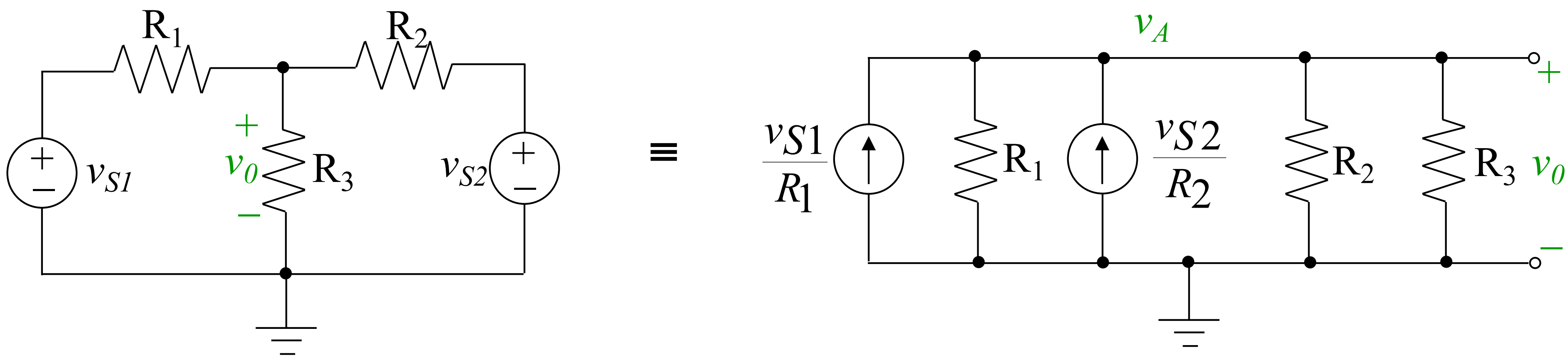


This is method 1 – transform the voltage sources

Applicable since voltage sources appear in series with Resist

Now use nodal analysis with one node, A

Nodal Analysis Ex. 3-4 (T&R, 5th ed, p. 76)



This is method 1 – transform the voltage sources

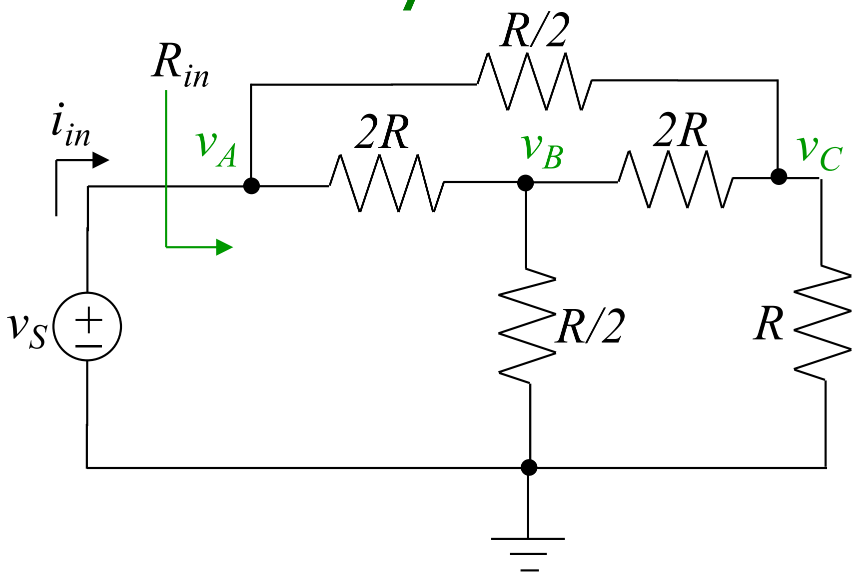
Applicable since voltage sources appear in series with Resist

Now use nodal analysis with one node, A

$$(G_1 + G_2 + G_3)v_A = G_1 v_{S1} + G_2 v_{S2}$$

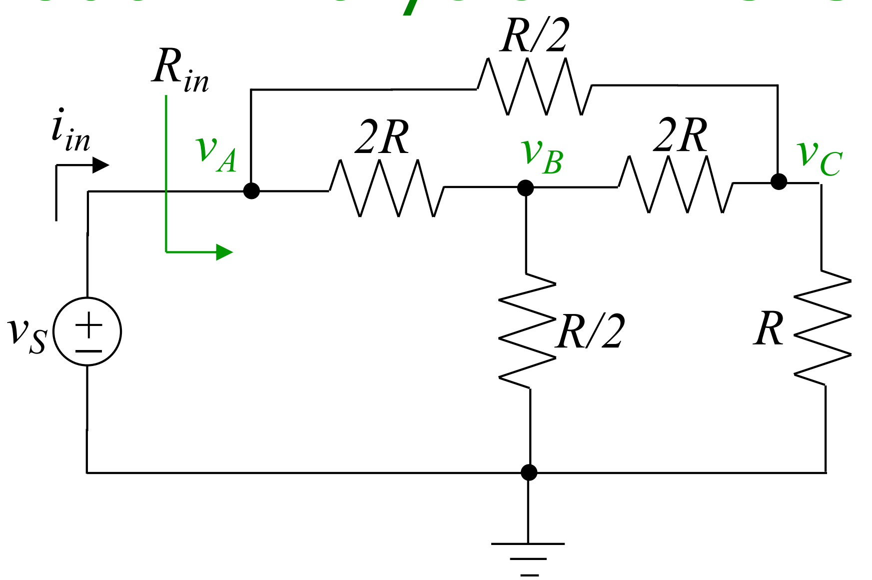
$$v_A = \frac{G_1 v_{S1} + G_2 v_{S2}}{G_1 + G_2 + G_3}$$

Nodal Analysis Ex. 3-5 (T&R, 5th ed, p. 77)



What is the circuit input resistance viewed through v_S?

Nodal Analysis Ex. 3-5 (T&R, 5th ed, p. 77)



What is the circuit input resistance viewed through v_S?

$$v_A = v_S$$

$$-0.5Gv_A + 3Gv_B - 0.5Gv_C = 0$$

$$-2Gv_A - 0.5Gv_B + 3.5v_C = 0$$

Rewrite in terms of v_S, v_B, v_C

This is method 2

$$3Gv_B - 0.5Gv_C \square 0.5Gv_S$$

$$-0.5Gv_B \square 3.5Gv_C \square 2Gv_S$$

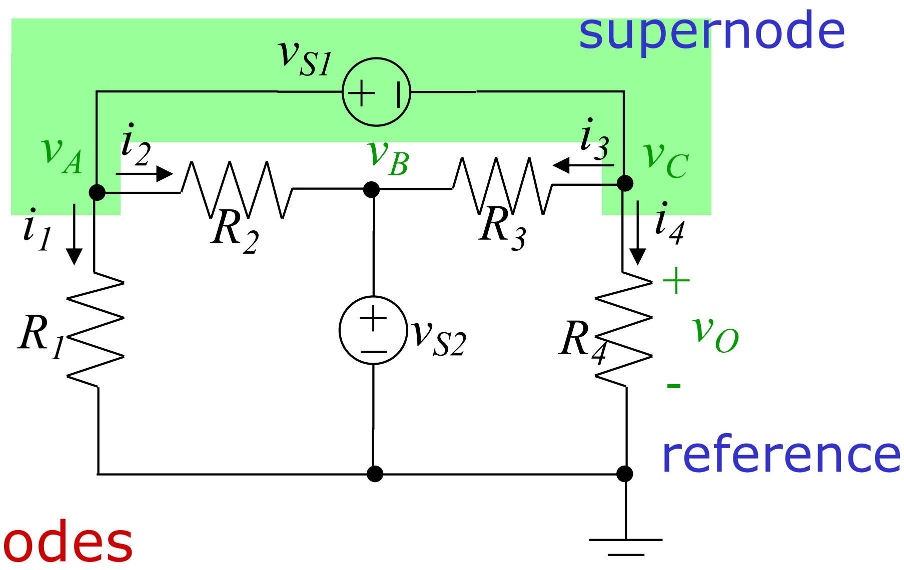
Solve

$$v_B = \frac{2.75v_S}{10.25}, \quad v_C = \frac{6.25v_S}{10.25}$$

$$i_{in} = \frac{v_S - v_B}{2R} + \frac{v_S - v_C}{R/2} = \frac{11.75v_S}{10.25R}$$

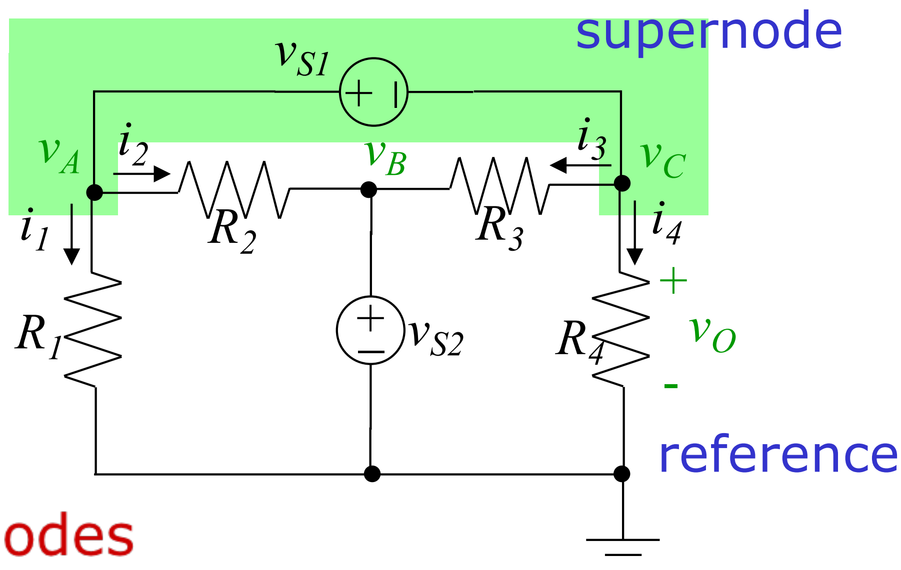
$$R_{in} = \frac{10.25R}{11.75} = 0.872R$$

Nodal Analysis Ex. 3-6 (T&R, 5th ed, p. 78)



Method 3 – supernodes

Nodal Analysis Ex. 3-6 (T&R, 5th ed, p. 78)



Method 3 – supernodes

KCL for supernode: $i_1 + i_2 + i_3 + i_4 = 0$

Or, using element equations

$$G_1 v_A + G_2 (v_A - v_B) + G_3 (v_C - v_B) + G_4 v_C = 0$$

Now use $v_B = v_{S2}$

$$(G_1 + G_2) v_A + (G_3 + G_4) v_C = (G_2 + G_3) v_{S2}$$

Other constituent relation

$$v_A - v_C = v_{S1}$$

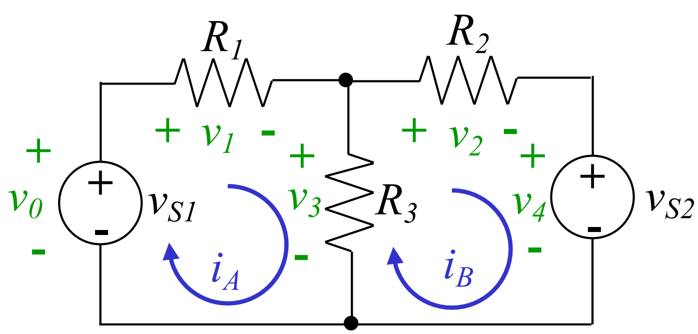
Two equations in two unknowns

Mesh Current Analysis

Dual of Nodal Voltage Analysis with KCL

- Mesh Current Analysis with KVL
  - Mesh = loop enclosing no elements
- Restricted to Planar Ccts – no crossovers (unless you are really clever)
- Key Idea: *If element K is contained in both mesh i and mesh j then its current is $i_k=i_i-i_j$ where we have taken the reference directions as appropriate*

Same old tricks you already know



Mesh A: $-v_0+v_1+v_3=0$ $\quad v_1=R_1i_A$ $\quad v_0=v_{S1}$

Mesh B: $-v_3+v_2+v_4=0$ $\quad v_2=R_2i_B$ $\quad v_4=v_{S2}$

$v_3=R_3(i_A-i_B)$

$$(R_1+R_3)i_A-R_3i_B=v_{S1}$$
$$-R_3i_A+(R_2+R_3)i_B=-v_{S2}$$

$$\begin{pmatrix} R_1+R_3 & -R_3 \\ -R_3 & R_2+R_3 \end{pmatrix}\begin{pmatrix} i_A \\ i_B \end{pmatrix}=\begin{pmatrix} v_{S1} \\ -v_{S2} \end{pmatrix}$$

Mesh Analysis by inspection $Ri = v_S$

Matrix of Resistances R

Diagonal ii elements: sum of resistances around loop

Off-diagonal ij elements: - resistance shared by loops i and j

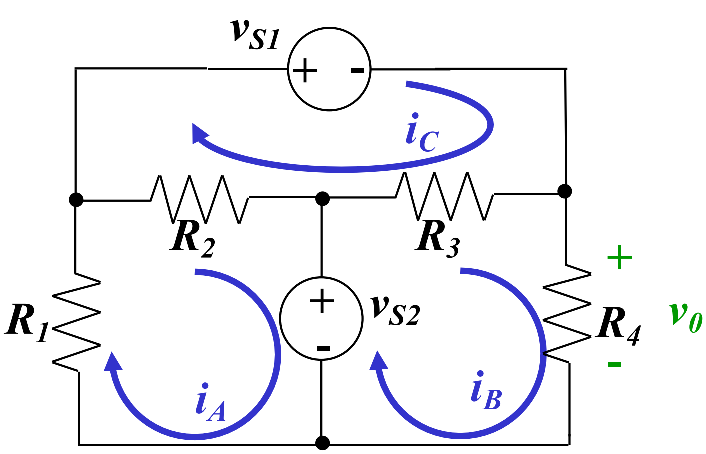
Vector of currents i

As defined by you on your mesh diagram

Voltage source vector v_S

Sum of voltage sources ***assisting*** the current in your mesh

If this is hard to fathom, go back to the basic KVL to sort these directions out



Mesh Analysis by inspection $Ri \,\square\, v_S$

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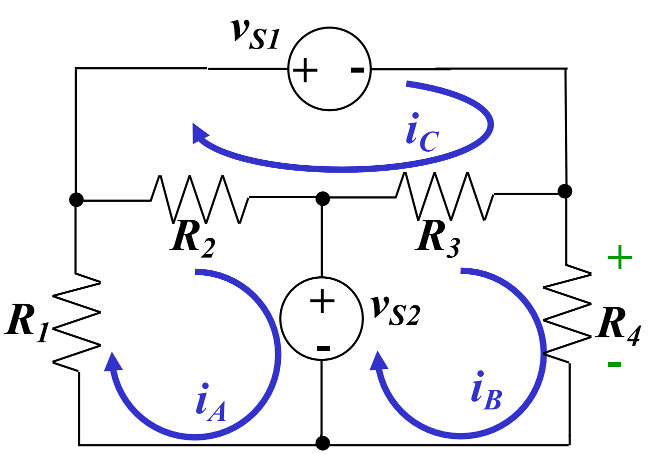


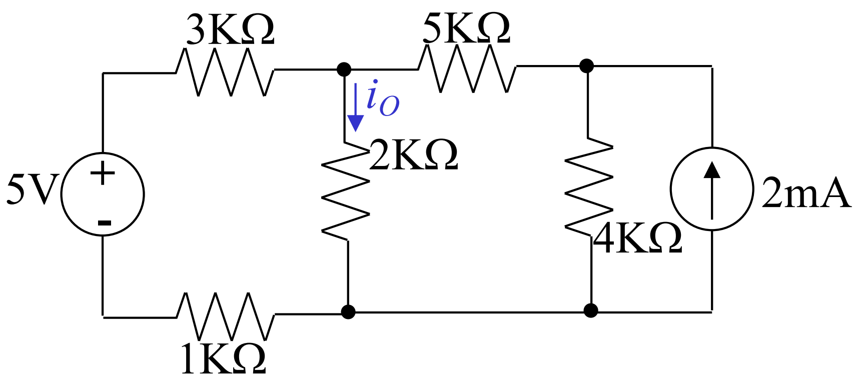
$$\begin{pmatrix} R_1+R_2 & 0 & -R_2 \\ 0 & R_3+R_4 & -R_3 \\ -R_2 & -R_3 & R_2+R_3 \end{pmatrix}\begin{pmatrix} i_A \\ i_B \\ i_C \end{pmatrix} = \begin{pmatrix} -v_{S2} \\ v_{S2} \\ -v_{S1} \end{pmatrix}$$

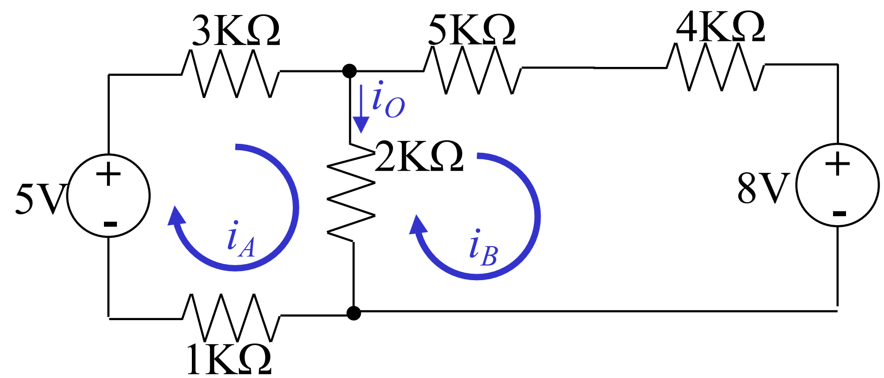
Mesh Equations with Current Sources

Duals of tricks for nodal analysis with voltage sources

1. Source transformation to equivalent

T&R, 5th ed, Example 3-8 p. 91





$$\begin{pmatrix} 6000 & -2000 \\ -2000 & 11000 \end{pmatrix}\begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$$

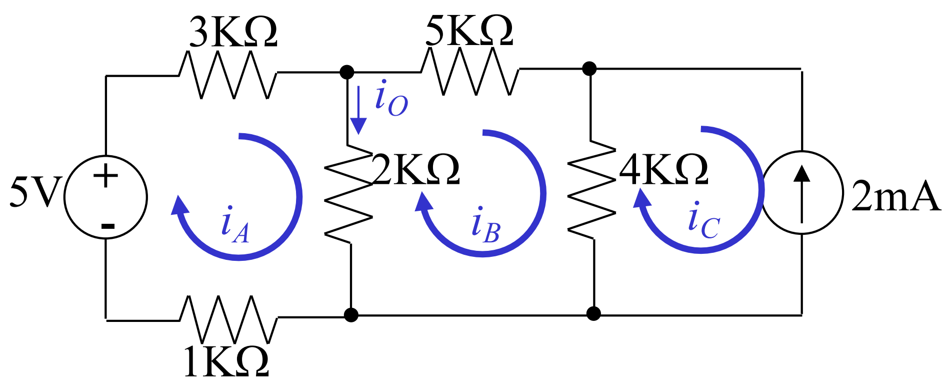
$i_A = 0.6290$ mA

$i_B = -0.6129$ mA

$i_O = i_A - i_B = 1.2419$ mA

Mesh Analysis with ICSs – method 2

Current source belongs to a single mesh



Same example

$$6000 i_A - 2000 i_B = 5$$

$$-2000 i_A + 11000 i_B - 4000 i_C = 0$$

$$i_C = -2\,\text{mA}$$

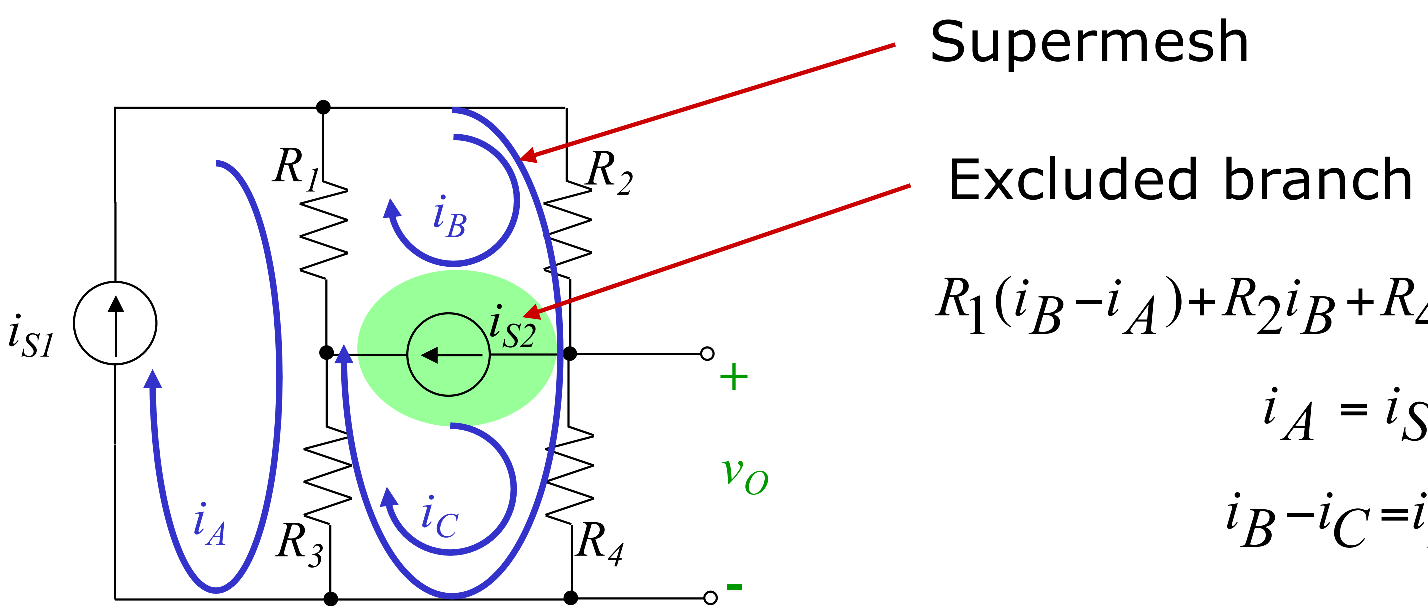
Same equations!
Same solution

Mesh Analysis with ICSs – Method 3

Supermeshes – easier than supernodes

Current source in more than one mesh and/or not in parallel with a resistance

1. Create a supermesh by eliminating the whole branch involved
2. Resolve the individual currents last

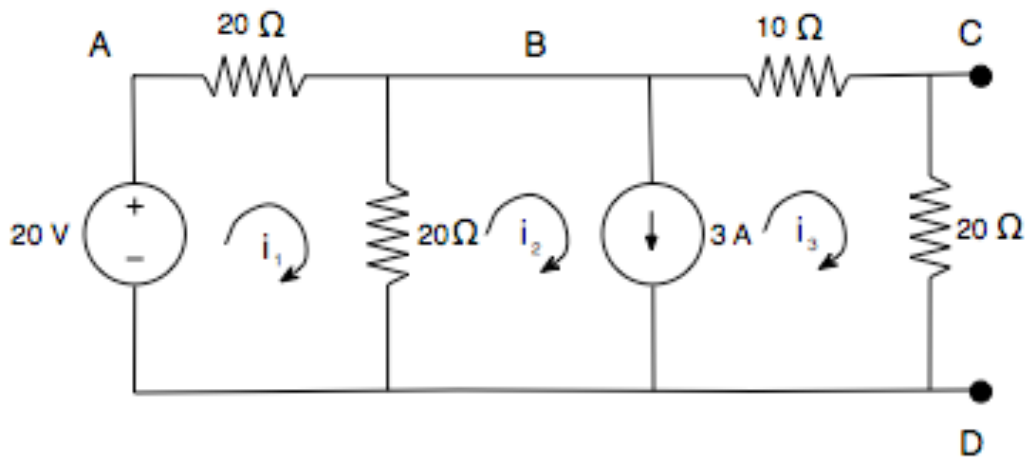


$$R_1(i_B - i_A) + R_2 i_B + R_4 i_C + R_3(i_C - i_A) = 0$$

$$i_A = i_{S1}$$

$$i_B - i_C = i_{S2}$$

Exercise from old midterm



Set up mesh analysis equations

Do not use any source transformation!

Supermesh

$$i_2 - i_3 = 3$$

$$10i_3 + 20i_3 + 20(i_2 - i_1) = 0$$

Remaining mesh

$$20i_1 + 20(i_1 - i_2) = 20$$

Summary of Mesh Analysis

1. Check if cct is planar or transformable to planar
2. Identify meshes, mesh currents & supermeshes
3. Simplify the cct where possible by combining elements in series or parallel
4. Write KVL for each mesh
5. Include expressions for ICSs
6. Solve for the mesh currents

Linearity & Superposition

Linear cct – modeled by linear elements and independent sources

Linear functions

Homogeneity: $f(Kx)=Kf(x)$

Additivity: $f(x+y)=f(x)+f(y)$

Superposition –follows from linearity/additivity

Linear cct response to multiple sources is the sum of the responses to each source

1. "Turn off" all independent sources except one and compute cct variables
2. Repeat for each independent source in turn
3. Total value of all cct variables is the sum of the values from all the individual sources

Superposition

Turning off sources

Voltage source

Turned off when $v=0$ for all i

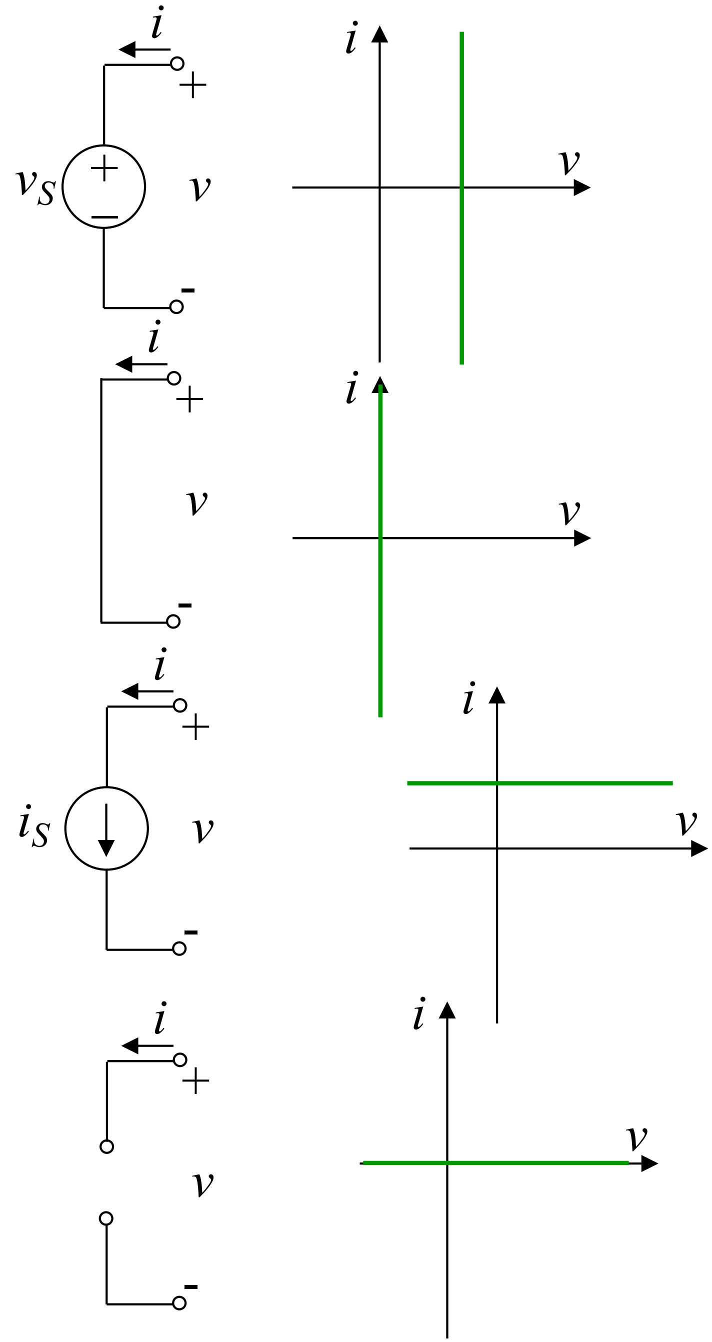
a short circuit

Current source

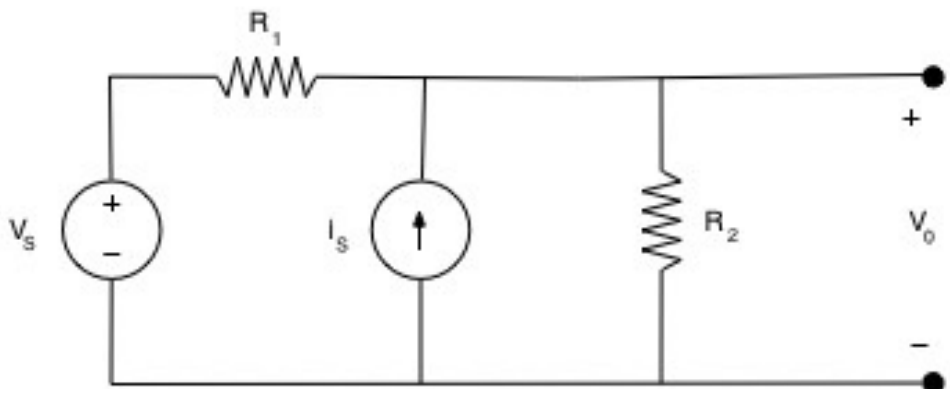
Turned off when $i=0$ for all v

an open circuit

<u>We have already used this in Thévenin and Norton equiv</u>

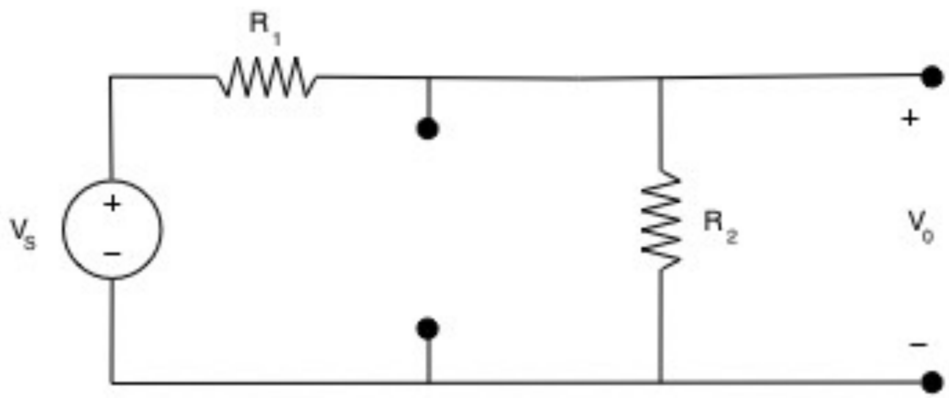


Superposition

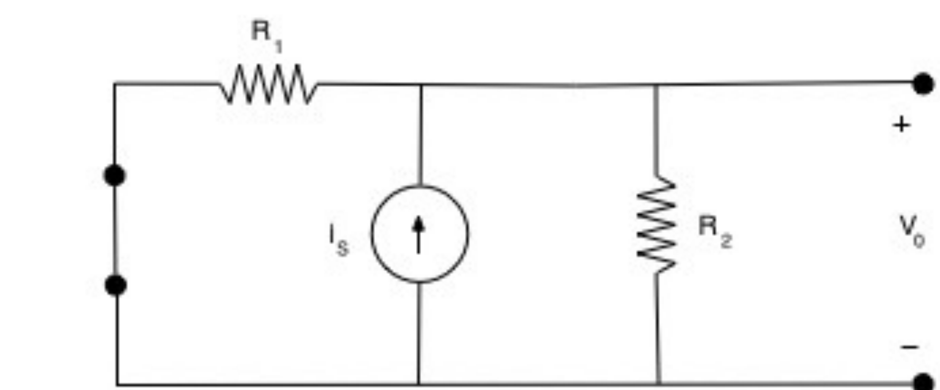


Find v_0 using superposition

$$v_0 = \frac{R_2}{R_1 + R_2} v_s + \frac{R_1 R_2}{R_1 + R_2} i_s$$



$$v_{01} = \frac{R_2}{R_1 + R_2} v_s$$



$$v_{02} = \frac{R_1 R_2}{R_1 + R_2} i_s$$

Where are we now?

Finished resistive ccts with ICS and IVS

- Two analysis techniques – nodal voltage and mesh current
  - Preference depends on simplicity of the case at hand
- The aim has been to develop general techniques for access to tools like matlab

Where to now?

- Active ccts with resistive elements – transistors, op-amps
  - Life starts to get interesting – getting closer to design
- Capacitance and inductance – dynamic ccts
  - Frequency response – s-domain analysis
- Filters