

Why Laplace transforms?

First-order RC cct

$$\text{KVL } v_S(t) - v_R(t) - v_C(t) = 0$$

instantaneous for each t

Substitute element relations

$$v_S(t) = V_A u(t), \quad v_R(t) = Ri(t), \quad i(t) = C \frac{dv_C(t)}{dt}$$

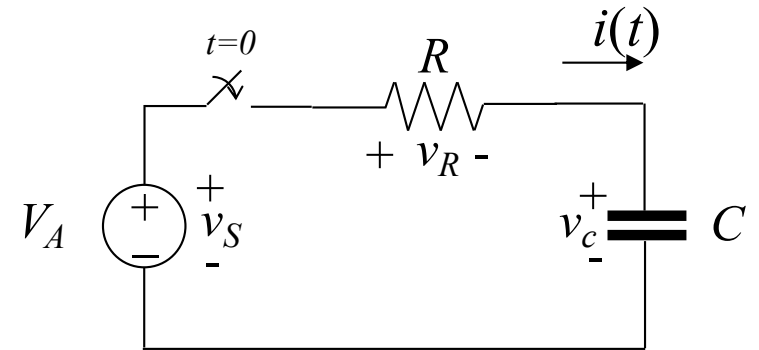
Ordinary differential equation in terms of capacitor voltage

$$RC \frac{dv_C(t)}{dt} + v_C(t) = V_A u(t)$$

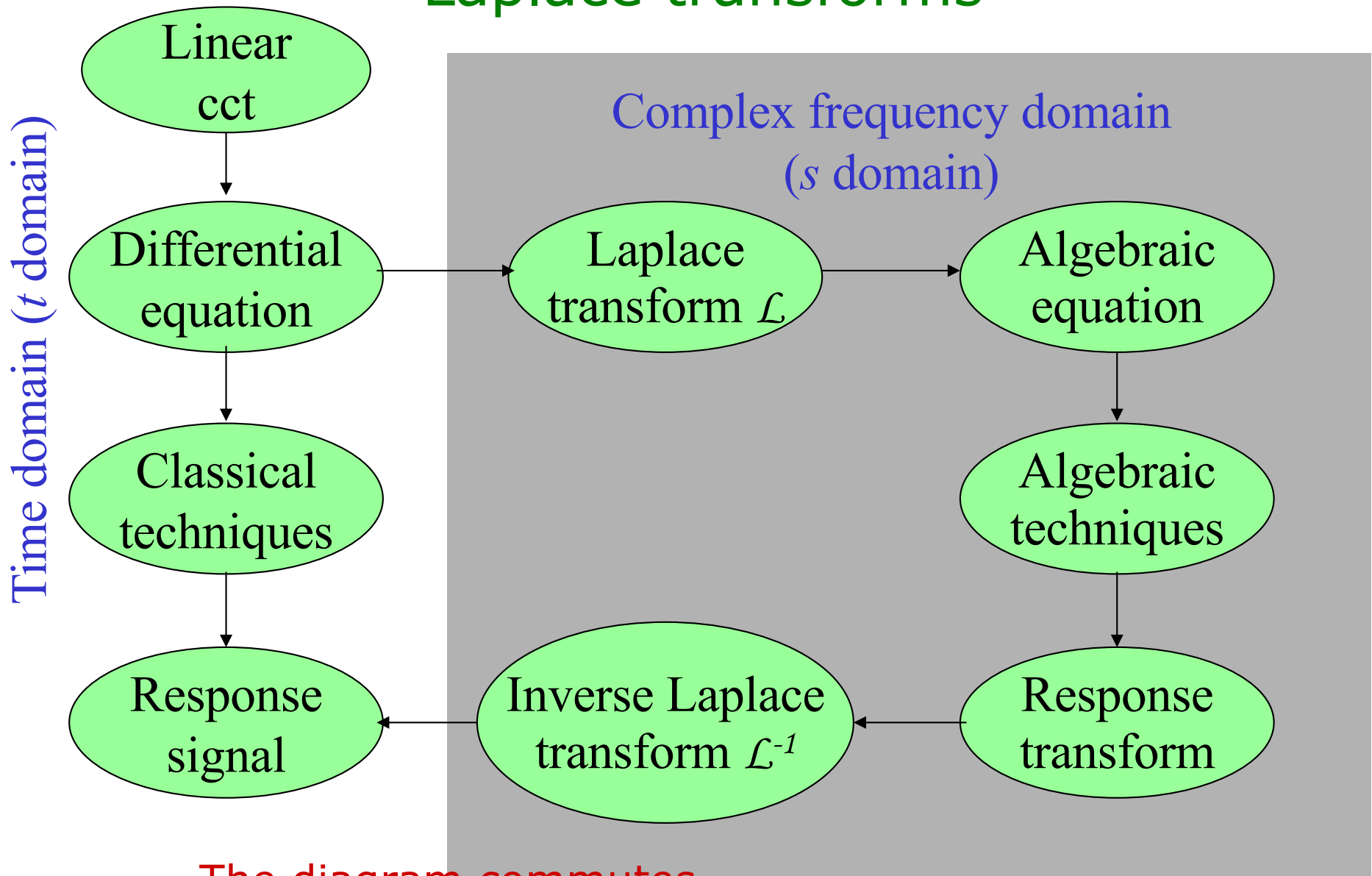
$$\text{Laplace transform } RC[sV_C(s) - v_C(0)] + V_C(s) = \frac{1}{s} V_A$$

$$\text{Solve } V_C(s) = \frac{V_A / RC}{s(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}$$

$$\text{Invert LT } v_C(t) = \left[V_A \left(1 - e^{-t/RC} \right) + v_C(0) e^{-t/RC} \right] u(t) \quad \text{Volts}$$



Laplace transforms



The diagram commutes

Same answer whichever way you go

Laplace Transform - definition

Function $f(t)$ of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

0^- limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

There is a need to worry about regions of convergence of the integral

Units of s are $\text{sec}^{-1} = \text{Hz}$

A frequency

If $f(t)$ is volts (amps) then $F(s)$ is volt-seconds (amp-seconds)

Laplace transform examples

Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925)

$$u(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$$

Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

Delta (impulse) function $\delta(t)$

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Delta (impulse) function $\delta(t)$

$$F(s) = \int_{0-}^{\infty} \delta(t)e^{-st} dt = 1 \text{ for all } s$$

Laplace Transform Pair Tables

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace Transform Properties

Linearity – absolutely critical property

Follows from the integral definition

$$\mathcal{L}\{Af_1(t) + Bf_2(t)\} = A\mathcal{L}\{f_1(t)\} + B\mathcal{L}\{f_2(t)\} = AF_1(s) + BF_2(s)$$

Example

$$\mathcal{L}(A\cos(\beta t)) =$$

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Example

$$\begin{aligned}\mathcal{L}(A \cos(\beta t)) &= \mathcal{L}\left(\frac{A}{2}(e^{j\beta t} + e^{-j\beta t})\right) = \frac{A}{2}\mathcal{L}(e^{j\beta t}) + \frac{A}{2}\mathcal{L}(e^{-j\beta t}) \\ &= \frac{A}{2} \frac{1}{s - j\beta} + \frac{A}{2} \frac{1}{s + j\beta} \\ &= \frac{As}{s^2 + \beta^2}\end{aligned}$$

Laplace Transform Properties

Integration property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^{\infty} \left[\int_0^t f(\tau)d\tau\right] e^{-st} dt$$

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$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^{\infty} \left[\int_0^t f(\tau)d\tau\right] e^{-st} dt$$

Denote
$$x = \frac{-e^{-st}}{s}, \text{ and } y = \int_0^t f(\tau)d\tau$$

so
$$\frac{dx}{dt} = e^{-st}, \text{ and } \frac{dy}{dt} = f(t)$$

Integrate by parts
$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \left[\frac{-e^{-st}}{s} \int_0^t f(\tau)d\tau\right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} f(t)e^{-st} dt$$

Laplace Transform Properties

Differentiation Property

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-)$$

Proof via integration by parts again

$$\begin{aligned}\mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[f(t)e^{-st} \right]_{0-}^{\infty} + s \int_{0-}^{\infty} f(t)e^{-st} dt \\ &= sF(s) - f(0-)\end{aligned}$$

Second derivative

$$\begin{aligned}\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} &= \mathcal{L}\left\{\frac{d}{dt}\left[\frac{df(t)}{dt}\right]\right\} = s\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df}{dt}(0-) \\ &= s^2 F(s) - sf(0-) - f'(0-)\end{aligned}$$

Laplace Transform Properties

General derivative formula

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \dots - f^{(m-1)}(0-)$$

Translation properties

s -domain translation

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s + \alpha)$$

t -domain translation

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as} F(s) \text{ for } a > 0$$

Laplace Transform Properties

Initial Value Property

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Final Value Property

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Caveats:

Laplace transform pairs do not always handle discontinuities properly

Often get the average value

Initial value property no good with impulses

Final value property no good with cos, sin etc

Rational Functions

We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in s

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$
$$= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

p_i are the poles and z_i are the zeros of the function

K is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

A Little Complex Analysis

We are dealing with linear ccts

Our Laplace Transforms will consist of rational functions (ratios of polynomials in s) and exponentials like $e^{-s\tau}$

These arise from

- discrete component relations of capacitors and inductors
- the kinds of input signals we apply
 - Steps, impulses, exponentials, sinusoids, delayed versions of functions

Rational functions have a finite set of discrete *poles*
 $e^{-s\tau}$ is an *entire function* and has no poles anywhere

To understand linear cct responses you need to look at the poles – they determine the exponential modes in the response circuit variables.

Two sources of poles: the cct – seen in the response to I_{cs}
the input signal LT poles – seen in the forced response

Residues at poles

Functions of a complex variable with isolated, finite order poles have *residues* at the poles

Simple pole: residue = $\lim_{s \rightarrow a} (s - a)F(s)$

Multiple pole: residue = $\frac{1}{(m-1)!} \lim_{s \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} ((s - a)^m F(s))$

Bundle complex conjugate pole pairs into second-order terms if you want

$$(s - \alpha - j\beta)(s - \alpha + j\beta) = \left[s^2 - 2\alpha s + (\alpha^2 + \beta^2) \right]$$

but you will need to be careful

Inverse Laplace Transform is a sum of complex exponentials
For circuits the answers will be real

Inverting Laplace Transforms in Practice

We have a table of inverse LTs

Write $F(s)$ as a partial fraction expansion

$$\begin{aligned} F(s) &= \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \\ &= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \\ &= \frac{\alpha_1}{(s - p_1)} + \frac{\alpha_2}{(s - p_2)} + \frac{\alpha_{31}}{(s - p_3)} + \frac{\alpha_{32}}{(s - p_3)^2} + \frac{\alpha_{33}}{(s - p_3)^3} + \dots + \frac{\alpha_q}{(s - p_q)} \end{aligned}$$

Now appeal to linearity to invert via the table

Surprise!

Computing the partial fraction expansion is best done by calculating the residues

Inverting Laplace Transforms

Compute residues at the poles $\lim_{s \rightarrow a} (s - a)F(s)$

$$\frac{1}{(m-1)!} \lim_{s \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^m F(s) \right]$$

Example $\frac{2s^2 + 5s}{(s+1)^3} = \frac{2(s+1)^2 + (s+1) - 3}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+1)^3}$

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$$\lim_{s \rightarrow -1} \frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} = -3 \qquad \lim_{s \rightarrow -1} \frac{d}{ds} \left[\frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} \right] = 1$$

$$\frac{1}{2!} \lim_{s \rightarrow -1} \frac{d^2}{ds^2} \left[\frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} \right] = 2$$

$$\mathcal{L}^{-1} \left[\frac{2s^2 + 5s}{(s+1)^3} \right] = e^{-t} (2 + t - 3t^2) u(t)$$

T&R, 5th ed, Example 9-12

Find the inverse LT of $F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)}$

T&R, 5th ed, Example 9-12

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$$F(s) = \frac{k_1}{s+1} + \frac{k_2}{s+1-j2} + \frac{k_2^*}{s+1+j2}$$

$$k_1 = \lim_{s \rightarrow -1} (s+1)F(s) = \left. \frac{20(s+3)}{s^2+2s+5} \right|_{s=-1} = 10$$

$$k_2 = \lim_{s \rightarrow -1+2j} (s+1-2j)F(s) = \left. \frac{20(s+3)}{(s+1)(s+1+2j)} \right|_{s=-1+2j} = -5-5j = 5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$f(t) = \left[10e^{-t} + 5\sqrt{2}e^{(-1+j2)t+j\frac{5}{4}\pi} + 5\sqrt{2}e^{(-1-j2)t-j\frac{5}{4}\pi} \right] u(t)$$
$$= \left[10e^{-t} + 10\sqrt{2}e^{-t} \cos\left(2t + \frac{5\pi}{4}\right) \right] u(t)$$

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

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Find the inverse LT of $F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$\begin{aligned} F(s) &= s + 2 + \frac{s + 2}{s^2 + 4s + 3} \\ &= s + 2 + \frac{0.5}{s + 1} + \frac{0.5}{s + 3} \end{aligned}$$

Invert as normal

$$f(t) = \left[\frac{d\delta(t)}{dt} + 2\delta(t) + 0.5e^{-t} + 0.5e^{-3t} \right] u(t)$$

Multiple Poles

Look for partial fraction decomposition

$$F(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)^2} = \frac{k_1}{s - p_1} + \frac{k_{21}}{s - p_2} + \frac{k_{22}}{(s - p_2)^2}$$

$$Ks - Kz_1 = k_1(s - p_2)^2 + k_{21}(s - p_1)(s - p_2) + k_{22}(s - p_1)$$

Equate like powers of s to find coefficients

$$k_1 + k_{21} = 0$$

$$-2k_1p_2 - 2k_{21}(p_1 + p_2) + k_{22} = K$$

$$k_1p_2^2 + k_{21}p_1p_2 - k_{22}p_1 = Kz_1$$

Solve

Recall motivating example for LT?

First-order RC cct

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instantaneous for each t

Substitute element relations

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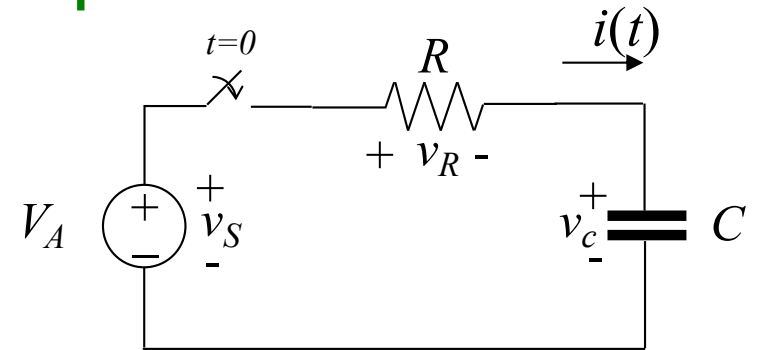
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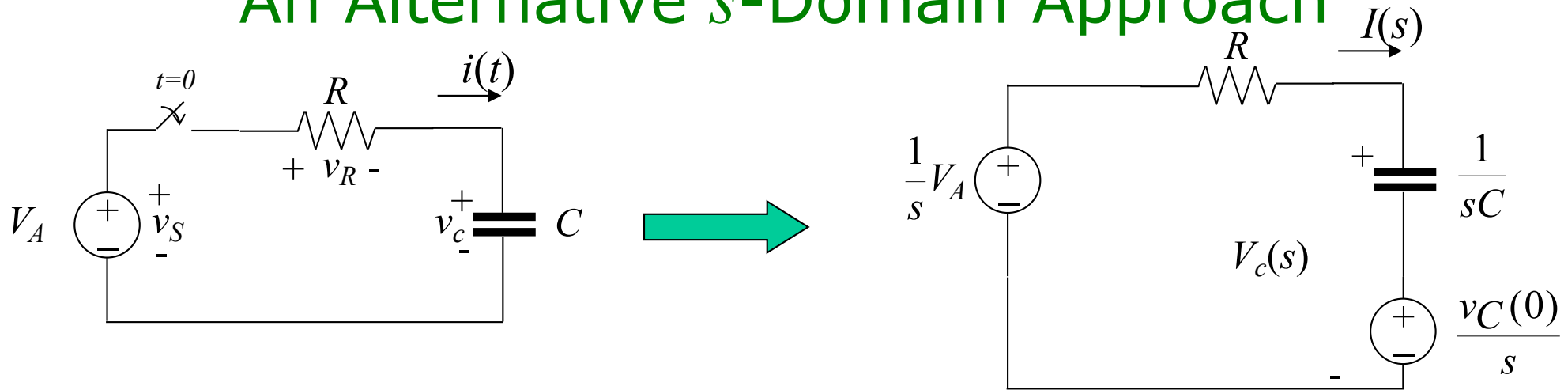
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$$\text{Solve } V_C(s) = \frac{V_A / RC}{s(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}$$

$$\text{Invert LT } v_C(t) = \left[V_A \left(1 - e^{-t/RC} \right) + v_C(0) e^{-t/RC} \right] u(t) \quad \text{Volts}$$



An Alternative s -Domain Approach



Transform the cct element relations

Work in s -domain directly

$$V_C(s) = \frac{1}{Cs} I_C(s) + \frac{v_C(0)}{s}$$

$$I_C(s) = sC V_C(s) - C v_C(0)$$

OK since \mathcal{L} is linear

Impedance + source

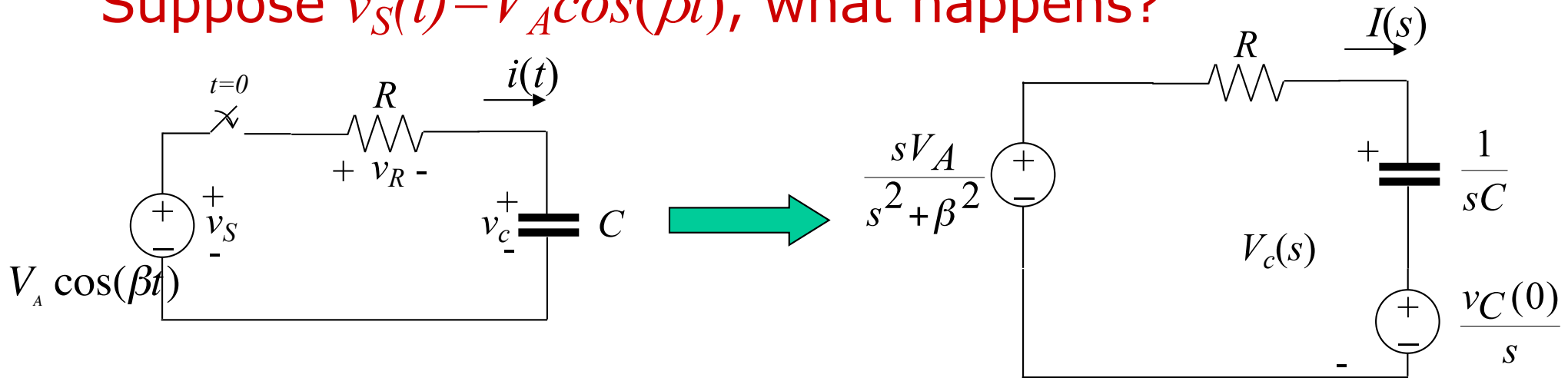
Admittance + source

KVL in s -Domain

$$sCRV_C(s) - CRv_C(0) + V_C(s) = \frac{1}{s}V_A$$

Time-varying inputs

Suppose $v_S(t) = V_A \cos(\beta t)$, what happens?



KVL as before $(RCs + 1)V_C(s) - RCv_C(0) = \frac{sV_A}{s^2 + \beta^2}$

$$V_C(s) = \frac{\frac{sV_A}{RC}}{(s^2 + \beta^2)(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}$$

Solve $v_C(t) = \left[\frac{V_A}{\sqrt{1 + (\beta RC)^2}} \cos(\beta t + \theta) - \frac{V_A}{1 + (\beta RC)^2} e^{-t/RC} + v_C(0) e^{-t/RC} \right] u(t)$